Wk 1

Semantics for quantifiers

1. Prove the following propositions:

a) $(\xi)\phi \vDash \neg(\exists\xi)\neg\phi$ and $\neg(\exists\xi)\neg\phi \vDash (\xi)\phi$ [3.6 E(b) in Bostock (1997), p. 97]

b) $\phi(\alpha/\xi) \vDash (\exists\xi)\phi$ [3.6 F(b)]

2. Show that the following formulæ are invalid and satisfiable:

a) (∃x)(y)(Pxy ⊃ Pyx)

b) ((x)Px ⊃ (x)Qx) ⊃ (x)(Px ⊃ Qx)

c) (∃x)(y)¬Pxy ⊃ (y)¬(∃x)Pxy

3. Find a counter-example to the sequent [3.10.1 in Bostock (1997), p. 138]:

(x)(y)(z)(Rxy & Ryz ⊃ Rxz) ⊨ (x)(y)(z)(¬Rxy & ¬Rxz ⊃ Rxz)

Wk 2

Further issues in first-order logic

1. Prove the following claim:

If $\vdash \phi$, then $\vdash \phi(\alpha/\xi)$, where α and ξ are any terms.

Hint: first prove the case where ϕ is an axiom.

2. Prove the following claim:

If D and D` are domains of the same cardinality, then a first-order formula $\phi$ is satisfiable in D just in case it is satisfiable in D`.

3. Past Papers, TT 99, #3.

Wk 3

Recursive functions

1. *Past Papers, TT 00, 13 (b)-(c)*

a) Assume that $k_1, \ldots, k_n$ is a sequence of distinct numbers and that $l_1, \ldots, l_n$ is any sequence of numbers. Show that if $f(x) = \begin{cases} l_i \text{ when } x = k_i \\ \text{undefined otherwise} \end{cases}$, then f is computable.

b) Conclude that every 1-place function with a finite domain is computable.

2. *TT 98, 14 (c)-(d)*

a) Define what is it for a relation to be primitive recursive.

b) Show that if $R(x,y)$ and $S(x,y)$ are primitive recursive relations, then so are the following: $\neg R(x,y)$, $R(x,y) \vee S(x,y)$, $R(x,y)$ & $S(x,y)$, $(\exists z < y)R(x,z)$, $(\forall z < y)R(x,z)$.

3. *TT 00, 14 (c)*

Assume that the functions f_1, f_2, g_1, and g_2 satisfy the following equations:

$f_1(0) = 0,\ f_2(0) = 0,\ f_1(x+1) = g_1(f_2(x)),\ f_2(x+1) = g_2(f_1(x)).$

Show that if g_1 and g_2 are primitive recursive, then so are f_1 and f_2.

4. What is the evidence that Church's thesis for predicates is true?

5. Describe a Turing machine that computes the function $f(x, y) = x + y$.

Wk 4

Recursive functions, undecidability &c.

1. Let $\mathfrak{L}$ be a formal language. Conclude that the procedure identifying the given finite string of symbols as a term or formula of $\mathfrak{L}$ is decidable.

2. *TT 00, 15 (a,b)*

Assume an exhaustive enumeration $\phi_0, \phi_1, ..., \phi_e, ...$ of one-place partial recursive functions and a three-place recursive relation T s.t. $\forall e \forall x(x \in \mathrm{Dom}(\phi_e) \leftrightarrow \exists y T(e,x,y))$.

A set S is *semi-recursive* iff there is a one-place partial recursive function f s.t. $\forall x(x \in S \leftrightarrow f(x) = 1)$.

a) Show that S is semi-recursive iff there is a one-place partial recursive function f s.t. $S = \mathrm{Dom}(f)$.

b) Show that S is semi-recursive iff there is a two-place recursive function R s.t. $\forall x(x \in S \leftrightarrow \exists y R(x,y))$.

3. *TT 99, 13 (b)*

Suppose that $\chi : \mathbb{N}^2 \to \mathbb{N}$ is primitive recursive. Let $h(n,m) \overset{df}{=} \sum_{i=0}^{m} i \cdot \chi(n,i)$.

Prove that h is primitive recursive.

4. *TT 99, 12 (a,c)*

a) Define what it is for a function of natural numbers to be representable in an axiomatic theory.

b) Prove that exponentiation on the natural numbers is representable in Peano Arithmetic.

Wk 5

Consistency, undecidability

1. Show that the relation $x *_p y = z$ is recursively enumerable for any prime *p*, where ‘*’ stands for concatenation.

2. *TT 00, 16 (c)*

Deduce that no consistent theory of arithmetic in which all recursive functions are representable is decidable.

3. *TT 99, 13 (a,b)*

a) Define ω-consistency for S a theory in a language with a numeral for each natural number, and prove that if S is ω-consistent, then it is consistent.

b) Show that for G the Gödel sentence of an axiomatic system S with arithmetized syntax, if S is ω-consistent, then $S \nvdash \neg G$. [You may take as given any standard properties of arithmetized syntax.]

4. Show that consistency is a strictly weaker condition than ω-consistency.

<u>*Readings*</u>

Enderton, *A Mathematical Introduction to Logic*, §§3.4-3.5.

Boolos and Jeffrey, *Computability and Logic*, ch.15.

Wk 6

Consistency, undecidability II

1. Prove that for any formal system of arithmetic, the condition of being ω-consistent is strictly weaker than that of being sound with respect to truth in arithmetic. [Hint: use the diagonal lemma.]

2. Let $\phi(x)$ be a recursively enumerable (r.e.) formula that expresses the set Φ of Gödel numbers of formulæ provable in P.A.

a) Show that for any r.e. sentence S, if S is true, then PA $\vdash S$, and conclude that for every r.e. sentence S, the sentence $S \supset \phi(\ulcorner S \urcorner)$ is true.

b) Show that for some sentence S, the sentence $S \supset \phi(\ulcorner S \urcorner)$ is false.

3. Show that if an axiomatic theory $\mathfrak{T}$ is ω-consistent, and S is any sentence in its language, then $\mathfrak{T} \cup \{S\}$ or $\mathfrak{T} \cup \{\neg S\}$ is ω-consistent.

4*. Show that there is no complete ω-consistent extension of P.A. containing a false sentence.

Wk 7

Provability

1. *TT 00, 17 (a,b,c)*

Let T be any theory of arithmetic (not necessarily consistent) in which there is a fixed point for each 1-place predicate, and in which $\vdash_T \neg(0=1)$. Let $\mathrm{Pr}_T(x)$ be any predicate meeting Hilbert-Bernays (H-B) adequacy conditions on a provability predicate.

a) State these conditions

b) State and prove Löb's theorem

c) Deduce Gödel's Second Incompleteness Theorem in the form: if T is consistent, then $\nvdash_T \neg\mathrm{Pr}_T(\ulcorner 0=1 \urcorner)$.

2. Let $\mathfrak{T}$ be an arithmetical theory supplied with the predicate B satisfying H-B conditions for the provability predicate. Suppose that X is a sentence of $\mathfrak{T}$ and $BX \supset X$ is provable in $\mathfrak{T}$ and that there is a sentence Y s.t. $Y \equiv (BY \supset X)$ is provable in $\mathfrak{T}$. Then X is provable in $\mathfrak{T}$.

3. *TT 00, 17 (a,b,c)*

Prove Löb's theorem from the Second Incompleteness Theorem. [Hint: prove the contrapositive of Löb's theorem by applying Second Incompleteness Theorem to the consistency of $T \cup \{\neg\phi\}$, where $T \nvdash \phi$.]

<u>*Readings*</u>

Boolos and Jeffrey, ch. 16;

Boolos, *The Logic of Provability* (1993), Introduction and ch. 3.

Wk 8

Provability II

1. *TT 98, 17 (a,b)*

a) Assume that $\mathrm{Pr}_T(x)$ be any predicate meeting Hilbert-Bernays (H-B) adequacy conditions on a provability predicate for a theory T. Show that $\neg\mathrm{Pr}_T(\ulcorner 0 = 1\urcorner)$ is provably equivalent in T to the Gödel sentence of T constructed using $\mathrm{Pr}_T(x)$.

b) Let $\mathrm{Prov}_T(x, y)$ be an arithmetical formula in the language of T, with no unbounded quantifiers, s.t. for formal numerals $\overline{m}$ and $\overline{n}$, $\mathrm{Prov}_T(\overline{m},\overline{n})$ is true iff m is the Gödel number of a proof in T of the formula whose Gödel number is n (compare Boolos and Jeffrey, ch. 16). Assume that T proves all those true arithmetical sentences in the language of T which contain no unbounded quantifiers. Let

$\mathrm{Pr}_T^*(x) \overset{def}{=} \exists y(\mathrm{Prov}_T(y,x) \,\&\, \forall z < y \neg\mathrm{Prov}_T(z,\mathrm{neg}(x))),$ where $\mathrm{neg}(x)$ is the Gödel number of the formula whose Gödel number is x. Show that $T \vdash \neg\mathrm{Pr}_T^*(\ulcorner 0 = 1\urcorner)$.

2. Let $\mathfrak{T}$ be a theory as in Wk 7, no. 2. Show that for any two sentences X and Y of $\mathfrak{T}$, the sentence $B(Y \equiv (BY \supset X)) \supset (B(BX \supset X) \supset BX)$ is provable in $\mathfrak{T}$. [Hint: first show that $B(Y \equiv (BY \supset X)) \supset (BY \supset BX)$ is provable.]