Formal Logic

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TT05

Contents

Week 1 Propositional calculus

1. Find a set Γ of formulae such that Γ is not simultaneously satisfiable, but for any two members X and Y of Γ, $\{X, Y\}$ is simultaneously satisfiable.

2. Given that Γ and Δ are simultaneously satisfiable, would $\Pi = \Gamma \cap \Delta$ be simultaneously satisfiable?

3. Prove the following results:

(a) $\{A \supset B, \neg A \supset B\} \vdash B$.

(b) $A \vee B \vdash B \vee A$.

(c) $\vdash \neg(A \wedge \neg A)$.

4. Find a disjunctive normal form for the formula $(B \wedge C) \supset (A \leftrightarrow (\neg B \vee C))$.

5. Show that the formula:

$$\left(\bigvee_{1 \leq i \leq n} A_i \right) \leftrightarrow \bigwedge_{1 \leq i \leq n} \left(\bigvee_{j \neq i} A_j \right)$$

is logically equivalent to the formula:

$$\bigwedge_{1 \leq i \leq n} \left(A_i \supset \bigvee_{j \neq i} A_j \right).$$

Readings

[Bos97] D. Bostock. *Intermediate Logic.* Oxford University Press, 1997, §§1.1-2.6, 5.1-5.4.

Week 2 Expressive adequacy and quantifiers

Definition. The binary connective $\oplus$ is *exclusive disjunction*, such that for any two formulae X and Y, $V(X \oplus Y) = \mathbf{T}$ iff $V(X) \neq V(Y)$.

Definition. An adequate set of connectives is *minimal* just in case no proper subset of it is an adequate set of connectives.

1. Determine whether the following sets of connectives are adequate and minimal:

(a) $\{\wedge, \vee, \supset\}$;

(b) $\{\oplus, \wedge, \top\}$;

(c) $\{\leftrightarrow, \vee, \bot\}$;

(d) $\{\supset, \bot\}$;

(e) $\{\wedge, \vee, \supset\}$;

(f) $\{\leftrightarrow, \oplus\}$.

2. Show that if a propositional formula A is satisfiable, then any substitution instance of A is also satisfiable.

3. Let the domain of quantification be the set of natural numbers. Let $S(x, y, z)$ be interpreted as $x + y = z$, and let $P(x, y, z)$ be interpreted as $x \cdot y = z$.

(a) Write down a formula with one free variable x which is true just in case $x = 0$;

(b) Write down a formula with one free variable x which is true just in case $x = 2$;

(c) Write down a formula with two free variables x and y which is true just in case $x = y$;

(d) Write down a formula with two free variables x and y which is true just in case $x \leq y$.

4. Show that the following formulae are invalid and satisfiable:

(a) $\exists x \forall y (Pxy \supset Pyx)$;

(b) $\forall x Px \supset \forall x Qx \supset \forall x (Px \supset Qx)$;

(c) $\exists x \forall y \neg Pxy \supset \forall y \neg \exists x Pxy$.

5. Consider the following formulae:

(a) $\exists x \forall y \exists z ((Px \supset Rxy) \wedge Py \wedge \neg Ryz)$;

(b) $\exists x \exists z ((Rzx \supset Rxz) \supset \forall y Rxy)$;

(c) $\forall x \forall y ((Px \wedge Rxy) \supset ((Py \wedge \neg Ryx) \supset \exists z (\neg Rzx \wedge \neg Ryz)))$.

Determine whether they are satisfied on **one** of the following models:

- The domain of quantification is $\mathbb{N}$, Rxy is interpreted as $x \leq y$, and Px is interpreted as 'x is an even integer';
- The domain is $\mathbb{R}$, the extension of R is the set of all pairs $\langle x, y \rangle \in \mathbb{R}^2$ such that $y = x^2$, and the extension of P is the subset of rational numbers.

Hint: Begin by transforming the formulae with the aid of distribution rules for quantifiers.

Readings

[Bos97] D. Bostock. *Intermediate Logic*. Oxford University Press, 1997, §§2.7-3.4.

Week 3 Predicate calculus

1. Let A be the following formula:

$$\forall x \forall y \forall z((\exists t(Rtx \wedge Rty) \wedge \exists t(Rty \wedge Rtz)) \supset \exists t \forall u(Rut \supset (Rux \wedge Ruz)))$$

and let the domain of quantification be the set M of natural numbers greater than 1 and the predicate Rxy be interpreted as 'x divides y'.

(a) Find a prenex normal form of A;

(b) Is A satisfied on M?

2. Find a prenex normal form of the formula:

$$\exists x \neg(\forall y Pxyz \supset \exists u Qxu) \wedge \forall t \neg \forall v(At \vee Bv).$$

3. Let A be a formula of predicate calculus with a fixed interpretation $\mathscr{M}$ of any cardinality whose predicate-letters are all one-place. Show that there is a quantifier-free formula equivalent to A over the same interpretation $\mathscr{M}$.

4. Show that any formula $F(x_1, \ldots, x_n)$ defined over a finite domain may be represented by a formula containing only one-place predicate-letters.

5. Show that the formula:

$$\exists x \forall y(Fxy \supset (\neg Fyx \supset (Fxx \leftrightarrow Fyy)))$$

is true on any interpretation whose domain contains no more than three individuals.

Readings

[Bos97] D. Bostock. *Intermediate Logic.* Oxford University Press, 1997, §§3.5-3.9.

Week 4 Axiomatic proofs

1. Prove the following meta-theorems:

(a) If $\Gamma \cup \{\neg A\} \vdash \neg B$, then $\Gamma \cup \{B\} \vdash A$;

(b) $\{\neg A, A\} \vdash B$;

(c) If $\Gamma \vdash \neg\neg A$, then $\Gamma \vdash A$;

(d) If $\Gamma \vdash A \supset B$, then $\Gamma \cup \Delta \vdash B$.

Hint: Use the Deduction Theorem.

2. Show that $\Gamma \vdash A$ just in case for some finite set $\Delta \subset \Gamma$, we have $\Delta \vdash A$.

3. Could a formula be an instance of both axiom-schemas (A1) and (A2) in [Bos97, 194]?

4. Give axiomatic proofs of the following results:

(a) $\{A \supset B, \neg A \supset B\} \vdash B$.

(b) $A \vee B \vdash B \vee A$.

(c) $\vdash \neg(A \wedge \neg A)$.

(d) $\neg\exists x \neg A^{x/u} \vdash \forall x A^{x/u}$.

(e) $\{\exists x A \supset B^{x/u}, A\} \vdash \exists x B^{x/u}$.

5. Show that if $B_1, \ldots, B_n \vdash A$ and the variable u has no occurrences in B_i for any i, then $B_1, \ldots, B_n \vdash \forall x A^{x/u}$.[1]

6. Show that if $\Gamma \vdash \exists x A^{x/u}$ and $\Gamma \cup \{A\} \vdash B$, where u has no occurrence in any formula of Γ or in B, then $\Gamma \vdash B$. *Hint:* First prove that if $\Gamma \vdash A$ and u has no occurrence in any formula of Γ, then $\Gamma \vdash \forall x A^{x/u}$.

Readings

[Bos97] D. Bostock. *Intermediate Logic.* Oxford University Press, 1997, §§1.3, 5.1-5.7.

[1] The notation x/u is analogous to the notation ξ/α in [Bos97].

Week 5 Sequent calculi

1. Given the rules for natural deduction in [Bos97, §7.2], justify the following rules:[2]

(a) $$\frac{\Gamma \vdash A \qquad \Delta \vdash (A \supset B)}{\Gamma, \Delta \vdash B}\ (\textsc{Cut})$$

(b) $$\frac{\{\Gamma, A, B\} \vdash C}{\{\Gamma, A \wedge B\} \vdash C}$$

(c) $$\frac{\{\Gamma, A \wedge B\} \vdash C}{\{\Gamma, A, B\} \vdash C}$$

(d) $$\frac{\{\Gamma, A\} \vdash C \qquad \{\Gamma, B\} \vdash C}{\{\Gamma, A \vee B\} \vdash C}$$

(e) $$\frac{(A_1 \wedge \cdots \wedge A_n) \supset B}{A_1 \wedge \cdots \wedge A_n \vdash B}$$

2. Let A and B be the formulae such that the sequent $A \vdash B$ is provable and the sequents $A \vdash$ and $\vdash B$ are unprovable. Show that there is a formula C containing only parameters common to A and B, such that the sequents $A \vdash C$ and $C \vdash B$ are provable. (The formula C is called an 'interpolation formula for the sequent $A \vdash C$'.)

3. Find interpolation formulae for the following sequents:

(a) $\neg(\neg Q \vee R) \vdash P \supset Q$;

(b) $\neg(P \supset \neg(Q \wedge S)) \vdash ((S \supset (P \supset R)) \supset R)$.

4. Let A be a formula of predicate calculus constructed out of atomic formulae and their negations with the aid of $\wedge$, $\vee$, and containing existential and universal quantifiers. Let A' be a formula resulting from A by replacing its atomic formulae with their negations, the connectives $\wedge$ and $\vee$ with $\vee$ and $\wedge$ respectively, and the existential and universal quantifiers with universal and existential quantifiers respectively. Assuming the standard structural rules of sequent calculus, show that $\vdash A' \leftrightarrow \neg A$.

5. Find proofs of the following sequents assuming (i) the rules of natural deduction and (ii) of the semantic tableaux:

(a) $\forall x Fx \supset \forall y Gy \vdash \exists x \forall y (Fx \supset Gy)$;

(b) $\forall x \exists y (Fx \supset Gy) \vdash \exists x Fx \supset \exists y Gy$;

(c) $\vdash \neg\exists x Fx \supset \forall x (Fx \supset Gx)$.

Readings

[Bos97] D. Bostock. *Intermediate Logic.* Oxford University Press, 1997, §§6.2, 7.1-7.5.

[2] Let the symbol '$\vdash$' stand for '$\Rightarrow$' in [Bos97].

Week 6 Completeness

1. Show that if $\vdash A \supset B$ and A and B share no sentence parameters, then either $\vdash A$ or $\vdash B$.

2. Using the previous exercise, formulate the interpolation lemma from Week 5, exercise 2, for propositional calculus. Prove it explicitly using completeness of the propositional calculus.

3. Show that propositional calculus is decidable.

4. State and prove the compactness theorem for propositional calculus.

5. Consider a first-order theory with equality (for instance, the theory determined by the axioms (1)-(7) in [Men64, 75]). Show that if A is an instance of any of its axioms, then A is logically valid.

6. Let T be a first-order theory, A a formula, and A' a universal closure of A. Show that:

(a) If $T \supset A$, then every model of T satisfies A';

(b) If $\vdash A$, then the universal closure of A' is semantically valid.

7. Show that if T has a model, then T is non-contradictory. *Hint:* Use the previous exercise.

Readings

[Bos97] D. Bostock. *Intermediate Logic.* Oxford University Press, 1997, §§4.6-4.8.
[Men64] E. Mendelson. *Introduction to Mathematical Logic.* Van Nostrand Company, 1964, §§2.2-2.5, 2.8.

Week 7 Completeness, model theory

Definition. A consistent set of formulae Γ is *saturated* if for every formula A with a free parameter u, there is a constant symbol c such that the formula $\exists x A^{x/u} \supset A^{c/u}$ is in Γ.

1. Show that if a set of formulae Γ is not satisfiable in any infinite domain, then there is an integer n such that for all $i > n$, Γ is not satisfiable in any domain of cardinality i. *Hint:* Use the compactness theorem.

2. Reflect on the possible significance of the notion of saturation in the proof of the completeness theorem.

3. Show that if Γ is a set of formulae and A is a formula true in every model of Γ, then $\Gamma \vdash A$. *Hint:* Use the completeness theorem.

4. Let Γ be a consistent set of formulae in a language L with no finite models. Show that Γ is complete if Γ is κ-categorical for any $\kappa \geq \text{card}(L)$. *Hint:* Use Skolem-Löwenheim theorem and Lemma in [Men64, 69].

5. It is a theorem of set theory that that there are non-denumerable sets. Explain how an apparent paradox will then result from Skolem-Löwenheim theorem, and how it may be solved. *Hint:* distinguish different uses of the universal quantifier.

6. Explain why, unlike the case of propositional calculus, decidability of predicate calculus does not follow from the completeness theorem.

7. Formulate Beth's definability theorem. Explain how the interpolation lemma from Week 5, exercise 2, may be used in proving Beth's theorem.

Readings

[Men64] E. Mendelson. *Introduction to Mathematical Logic.* Van Nostrand Company, 1964, §§2.8, 2.11, 4.3.
[BJ89] G. Boolos and R. Jeffrey. *Computability and Logic.* Routledge & Kegan Paul, 1989, chs. 10, 24.

Week 8 Turing machines

Note The notation below differs slightly from the notation in [Men64, BJ89]. Thus, e.g., the command $q_1 0 \mapsto q_2 0 R$ below will correspond to the quadruple $q_1 0 R q_2$ in [Men64], while $a = S_0$.

1. Identify the function computed by the following Turing machine:

$$
\begin{array}{ll}
q_1 0 \mapsto q_2 0 R & q_1 1 \mapsto q_0 1 \\
q_2 0 \mapsto q_0 1 & q_2 1 \mapsto q_2 1 R \\
q_3 0 \mapsto q_0 0 & q_3 1 \mapsto q_3 1 L.
\end{array}
$$

2. Find a Turing machine computing the function $f(x) = 0$ and write it down in a sequence of commands.

3. Verify that the following Turing machine computes the function $f(x) = 2x$:

$$
\begin{array}{lll}
q_1 1 \mapsto q_3 0 & q_3 0 \mapsto q_3 R & q_3 1 \mapsto q_2 0 \\
q_2 0 \mapsto q_2 L & q_2 a \mapsto q_3 0 & q_3 a \mapsto q_4 L \\
q_4 0 \mapsto q_4 1 & q_4 1 \mapsto q_4 L & q_4 a \mapsto q_0 a.
\end{array}
$$

4. Find Turing machines for the functions $f(x) = x \dot{-} 1$ and $f(x) = \mathrm{sg}(x)$ defined in [BJ89, 84-5].

5. *If time permits,* verify that the following functions are primitively recursive:

(a) $f(x, y) = x + y$;

(b) $f(x, y) = x \cdot y$;

(c) $f(x, y) = x^y$;

(d) $f(x) = x!$.

Readings

[Men64] E. Mendelson. *Introduction to Mathematical Logic.* Van Nostrand Company, 1964, §§2.3, 5.2.
[BJ89] G. Boolos and R. Jeffrey. *Computability and Logic.* Routledge & Kegan Paul, 1989, chs. 3, 7.