Intermediate Logic

Lecture notes

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Chapter 11

Exercises

11.1 Revision: paraphrase etc.

Your answers must be as detailed as possible. Simple yes/no answers do not count.

11.1.1. Give the best paraphrases of the following sentences into the language of propositional logic:

1. Obama and Palin will not both win the election.
2. Either Obama will win the election and the world be saved or else he will not and Palin will triumph.
3. Nadal will win the next Grand Slam unless Federer improves.
4. If the government either prints money and creates inflation or improves the business environment, unemployment will begin to drop.
5. Barack will marry Sarah in January if he gets the divorce from Michelle in December, provided they don't kill each other before and the Democrats lose the election.
6. Barack will never marry Sarah, even if he falls in love with her.

Solution.

1. $\neg(P \wedge Q)$
2. $[(P \wedge Q) \wedge \neg R] \vee (\neg P \wedge R)$
3. $P \leftrightarrow \neg Q$
4. $[(P \wedge Q) \vee R] \supset S$
5. $(P \wedge Q) \supset (R \supset S)$
6. P

11.1.2. Could there be valid arguments under the following conditions:

1. Premisses all true, conclusion false.
2. Premisses all untrue, conclusion untrue.
3. Some premisses true, some untrue, conclusion untrue.
4. Premisses all true, negation of conclusion true.

Solution.

1. No.
2. Yes.
3. Yes.

4. No.

11.1.3. Establish whether the following arguments are valid:

1. Premisses: ‘Federer is tall’, ‘If Federer is tall, he is handsome’, ‘Federer is short’. Conclusion: ‘Federer is handsome.’
2. Premisses: ‘Federer is tall’, ‘If Federer is tall, he is handsome’. Conclusion: ‘$2 + 2 = 4$.’
3. Premisses: ‘Federer is tall’, ‘If Federer is tall, he is handsome’. Conclusion: ‘Federer is tall.’
4. Premisses: ‘Federer is tall’, ‘If Federer is tall, he is handsome’. Conclusion: ‘Federer is short.’

Solution.

1. Valid.
2. Valid.
3. Valid.
4. Invalid.

11.2 Paraphrase/interpretation

11.2.1. Paraphrase the following sentences into a predicate language. Explain the difficulties. Indicate the interpretation of predicates and individual parameters and specify the domain for successful paraphrases.

1. Only you can help me.
2. If Harry Potter gets angry at you, you can only pray to God.
3. Boys love girls, but girls love their mothers.
4. Bill Gates is the richest man in the world.
5. Two soldiers died in that battle.
6. There are at most two students in this class.
7. There are exactly two students in this class.

Solution.

1. $\forall x(Hxa \supset x = b)$, domain: all people.
2. $Aha \supset \forall x(Dax \supset x = p)$, domain: everything.
3. $\forall x[Bx \supset \exists y(Gy \wedge Lxy)] \wedge \forall x[Gx \supset \exists y(Myx \wedge Lxy)]$, domain: all people.
4. $\neg\exists x(Mx \wedge Rxg)$, domain: people in the world (on Earth?).
5. $\exists x\exists y(Sx \wedge Sy \wedge Dxb \wedge Dyb \wedge x \neq y)$, domain: people and battles.
6. We deny that there are (at least) three students:
   $\neg\exists x\exists y\exists z(Sxc \wedge Syc \wedge Szc \wedge x \neq y \wedge x \neq z \wedge y \neq z)$, domain: people and classes.
7. $\exists x\exists y(Sxc \wedge Syc \wedge x \neq y \wedge \forall z(Szc \supset (x = z \vee y = z))$, domain: people and classes.

11.2.2. By finding relevant interpretations, establish whether the following formulae are valid/invalid and satisfiable/unsatisfiable:

1. $\forall x(Px \supset Qx) \vee \neg(\forall xPx \supset \forall xQx)$
2. $\forall y\exists x(Pxy \vee Pyx) \supset (\forall x\exists yPxy \vee \forall x\exists yPyx)$
3. $\forall y(\exists zPyz \supset \exists zPzy) \supset \forall y(\forall zPyz \supset \forall zPzy)$

Solution.

1. Invalid and satisfiable. The formula is false on the left-hand interpretation and true on the right-hand one:

$M : \{1, 2, 4, 6 \ldots\}$	$M : \{0, 2, 4, \ldots\}$
$P :$ ① is even	$P :$ ① is odd
$Q :$ ① is odd	$Q :$ ① is even.

2. Invalid and satisfiable. The formula is false on the left-hand interpretation and true on the right-hand one:

$M : \{0, 1, 2\}$	$M : \{0, 1, 2\}$
$P :$ ① > ②	$P :$ ① = ②.

3. Invalid and satisfiable. Note that $\neg\forall y(\forall zPyz \supset \forall zPzy) \simeq \exists y(\forall zPyz \wedge \exists w\neg Pwy)$. The formula is false on the left-hand interpretation and true on the right-hand one:

$M : \{0, 1, 2, \ldots\}$	$M : \{0, 1, 2, \ldots\}$
$P :$ ① ≤ ②	$P :$ ① < ②.

11.3 General concepts

11.3.1. Accept or reject with a short explanation the following claims:

1. $X \subseteq X$
2. $X \subset X$
3. $\{a, b\} \in \{a, b, c\}$

Solution. Easy.

11.3.2. Examine whether $|X| = |Y|$:

1. $X = \{0, 1, 2, 3\}$, $Y = \{5, 1, 7, 8, 5\}$
2. $X = \{0, 1, 2, \ldots\}$, $Y = \{-1, 0, 1, 2 \ldots\}$
3. $X = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $Y = \{\ldots, -5, -3, -1, 1, 3, 5 \ldots\}$

Solution.

1. Easy.
2. Easy.
3. The bijective mapping is $f(n) = 2n - 1$.

11.3.3. Insert missing quotation marks in the following passage:

> 'You are sad,' the Knight said in an anxious tone: 'let me sing you a song to comfort you.'
>
> 'Is it very long?' Alice asked, for she had heard a good deal of poetry that day.
>
> 'It's long,' said the Knight, 'but very, very beautiful. Everybody that hears me sing it – either it brings the tears into their eyes, or else – '
>
> 'Or else what?' said Alice, for the Knight had made a sudden pause.
>
> 'Or else it doesn't, you know. The name of the song is called Haddocks' Eyes.'
>
> 'Oh, that's the name of the song, is it?' Alice said, trying to feel interested.
>
> 'No, you don't understand,' the Knight said, looking a little vexed. 'That's what the name is called. The name really is The Aged Aged Man.'
>
> 'Then I ought to have said That's what the song is called?' Alice corrected herself.
>
> 'No, you oughtn't: that's quite another thing! The song is called Ways and Means: but that's only what it's called, you know!'
>
> 'Well, what is the song, then?' said Alice, who was by this time completely bewildered.
>
> 'I was coming to that,' the Knight said. 'The song really is A-sitting On A Gate: and the tune's my own invention.'

What is the song called? (Give a full answer.)
Solution. 'The Aged Aged Man' (the name), 'Ways and Means' (the nickname), and possibly also 'A-sitting On A Gate'.

11.3.4. Is $(P_1 \supset P_2)$ a formula of $\mathsf{H_s}$, or is '$(P_1 \supset P_2)$' a formula of $\mathsf{H_s}$?
Solution. $(P_1 \supset P_2)$ is a formula of $\mathsf{H_s}$, but is '$(P_1 \supset P_2)$' is a name of the formula of $\mathsf{H_s}$.

11.3.5. Determine whether the formula $\neg A \supset (A \neg \supset B)$ is a formula of $\mathsf{H_s}$.
Solution. Easy.

11.3.6. Four students, Abe, Ben, Chris, and Dave, competed in a table-tennis tournament. When a journalist asked them which places they got, he received three different answers:

1. Chris was first, Ben second.
2. Chris was second, Dave third.

3. Abe was second, Dave fourth.

In each of the answers at least one part is correct. None of the students got the same place. Determine who got which place.
Solution. We need to know in what situation the sentence $(c_1 \vee b_2) \wedge (c_2 \vee d_3) \wedge (a_2 \vee d_4)$ is true, given the assumptions (e.g. that there can't be two people getting to the same place). Examining the truth-table of our sentence, we see that it is true when c_1, a_2, d_3 are true. Consequently, Chris was first, Abe second, Dave third, and Ben fourth.

11.4 Sentence calculus

11.4.1. Formulate the rules for building syntactic formation trees for the formulae of $\mathsf{H_s}$ (of the kind we discussed at the lecture).
Solution. Let $\mathscr{T}$ be a formation tree for A. Then $\mathscr{T}$ is an ordered dyadic tree whose points are formulae and whose origin is A. The following conditions are imposed on $\mathscr{T}$:

1. Each end point is a sentence parameter.
2. Each simple point has the form $\neg B$, and its only successor is B.
3. Each junction point has the form $B \triangle C$, whose successors are B and C (where $\triangle$ is one of the sentential connectives excluding negation).

11.4.2. Show that:

1. If $\Gamma \cup \{\neg A\} \vdash \neg B$, then $\Gamma \cup \{B\} \vdash A$.
2. If $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$.
3. If $\Gamma \vdash A$ and $\Delta \cup \{A\} \vdash B$, then $\Gamma \cup \Delta \vdash B$.

Solution.

1. Using the Deduction theorem:

$$\begin{aligned}
&\Gamma \cup \{\neg A\} \vdash \neg B \\
&\Gamma \vdash \neg A \supset \neg B \\
&\vdash \neg A \supset \neg B \supset (B \supset A) \\
&\Gamma \vdash B \supset A \\
&\Gamma \cup \{B\} \vdash A
\end{aligned}$$

2. By the definition of deduction.
3. Using the definition of deduction and Ex. 2.

11.4.3. Are the following sets of sentences simultaneously satisfiable:

1. Mary is a woman. Mary has forty biological children.
2. Mary is a woman. Mary has three billion biological children.
3. If Padua is where I think it is, Milan is west of it. Milan is not where I think it is, but Padua is. Milan is north west of Padua.
4. Padua is in Italy or Greece. If Padua is in Greece, Italy is not in Europe. Italy is in Europe. Padua is not in Europe.
5. If Padua is in Europe, it is in Italy. Padua is in Italy, but not in Europe.

Solution.

1. Simultaneously satisfiable.
2. Controversial.
3. $X = \{P \supset Q, P \wedge \neg R, \neg Q\}$: not simultaneously satisfiable.
4. $X = \{P \vee Q, Q \supset \neg R, R, \neg S\}$. But one might also assume that $P \supset S$ (why?). Then we would have $X' = \{P \vee Q, Q \supset \neg R, R, \neg S, P \supset S\}$. X' is not simultaneously satisfiable, but X is.
5. $X = \{P \supset Q, Q \wedge \neg P\}$: simultaneously satisfiable.

11.4.4. Find a disjunctive normal form for the formula $(Q \wedge R) \supset (P \leftrightarrow (\neg Q \vee R))$.
Solution. $P \vee \neg Q \vee \neg R$.

11.5 Predicate calculus

11.5.1. Which of the following expressions are predicates:

1. 3 is a divisor of x.
2. $y = x^2$.
3. $x^2 + x + 1 = 0$.
4. $x^2 + 2x + 1$.
5. x and y.

Solution.

1. A unary predicate.
2. A binary predicate.
3. A unary predicate.
4. Not a predicate.
5. Not a predicate.

11.5.2. How many free variables are in the following expressions:

1. $x^2 + y^2 = z^2$.
2. For every x there is y such that $x + y = 1$.
3. There is z such that for every x we have $x + y < z$.

Solution.

1. Three.
2. None.
3. One.

11.5.3. Let $\mathfrak{M} = \left\langle M; \overline{P^2} \right\rangle$, where M is the set of all subsets of some set X (that is, $M = \wp(X)$), P^2 is the predicate '①$\subseteq$②'. Write down a formula of $\mathsf{T_p}$ saying that:

1. x is the intersection of y and z.
2. x is the union of y and z.
3. $x = \varnothing$.
4. $x = X$.

Solution.

1. $Pxy \wedge Pxz \wedge \forall u((Puy \wedge Puz) \supset Pux)$.
2. $Pyx \wedge Pzx \wedge \forall u((Pyu \wedge Pzu) \supset Pxu)$.
3. $\forall y Pxy$.
4. $\forall y Pyx$.

11.6 Predicate calculus II

11.6.1. Let $\mathfrak{M} = \left\langle \{x \mid x \text{ is a Turkish city}\}; \overline{P^0}, \overline{Q^1}, \overline{R^2} \right\rangle$, where:

$$P: \text{2 plus 2 equals 4}$$
$$Q: \text{① is established after 1900}$$
$$R: \text{① is more ancient than ②.}$$

Determine for which of the following claims $\mathfrak{M}$ provides a counterexample:

1. $\vDash P$
2. $\vDash \forall x \exists y(\neg Qy \wedge Rxy)$
3. $\forall x Qx \vDash \exists x(Qx \wedge \exists y Rxy)$.

Solution.

1. No: P is valid.
2. Yes: consider the case when x is the most ancient city of Turkey.
3. No: $\forall x Qx$ is not true on $\mathfrak{M}$, so it is impossible to test the validity of the entailment on $\mathfrak{M}$.

11.6.2. Let A be an atomic sentence. Using semantic tableaux, examine the following claims:

1. $\forall x Px \vee A \vDash \forall x(Px \vee A)$.
2. $\exists x \forall y(Pxy \leftrightarrow \forall z((Pyz \wedge Pzy) \supset A)) \vDash A$.
3. $\forall x \exists y(Pax \wedge Qay) \vDash \exists y \forall x(Pax \wedge Qay)$.

Solution. See Figure **??**.

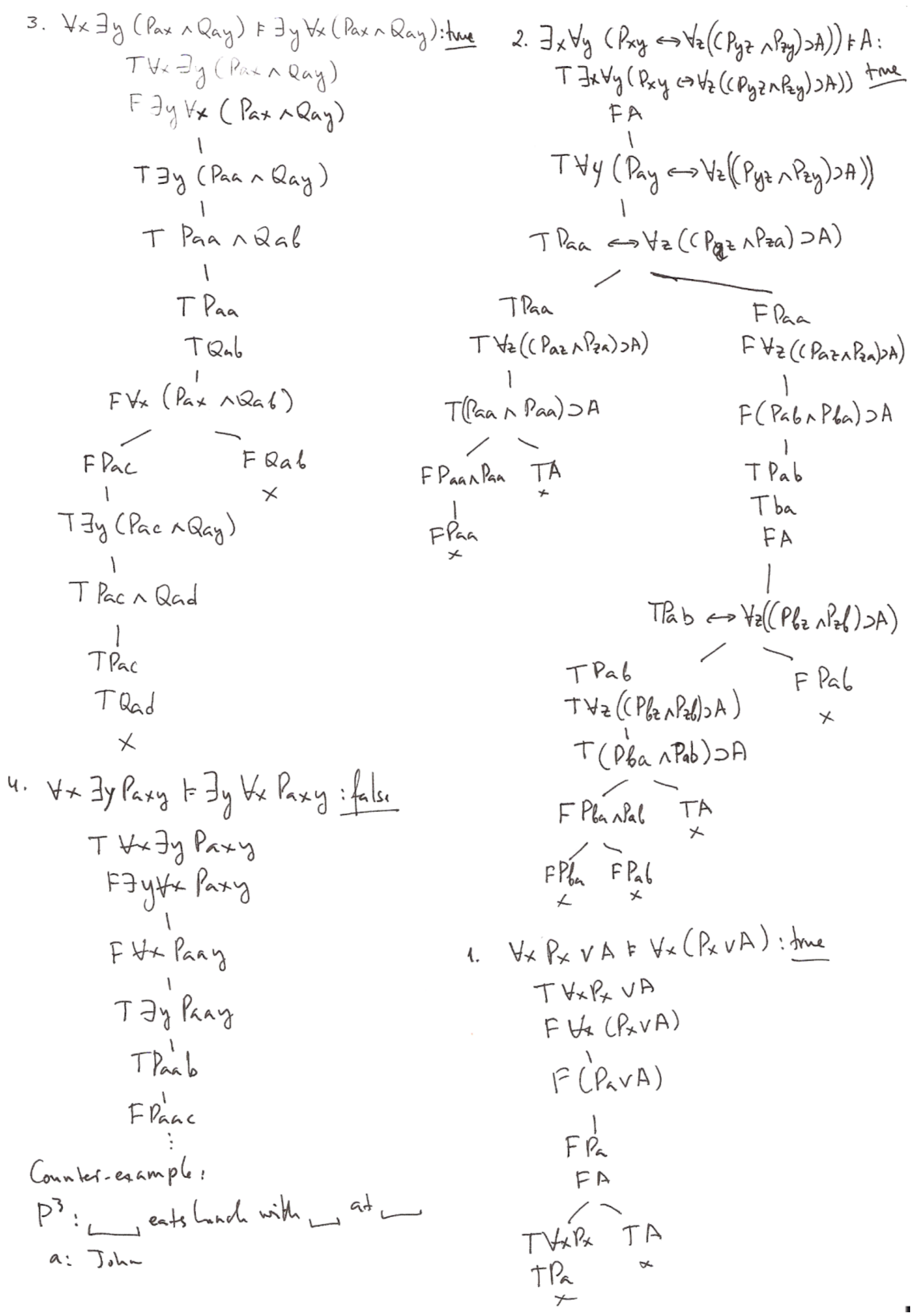

Figure 11.1: Solutions for Exercise ??.

11.7 Predicate calculus III

11.7.1. Using the tableau method, show that the following holds (where $A \dashv\vdash B$ iff $A \vdash B$ and $B \vdash A$):

1. $\forall x Px \dashv\vdash \neg\exists x \neg Px$
2. $\exists x Px \dashv\vdash \neg\forall x \neg Px$
3. $\forall x(\exists y Pxy \supset Qx) \dashv\vdash \forall x \forall y(Pxy \supset Qx)$
4. $\exists y \forall x(Pax \wedge Qay) \vdash \forall x \exists y(Pax \wedge Qay)$.

11.7.2. Find Skolem forms for the formulae:

1. $\exists x \forall y \exists z \forall v Rxyzv$;
2. $\forall x \exists y \forall v \exists z Rxyzv$.

11.8 Predicate calculus IV

11.8.1. Using the tableau method, examine the following claims:

1. $\forall x \forall y \forall z(Rxy \wedge Rxz \supset \neg Rxz) \vdash \forall x \neg Rxx$
2. $\forall x \exists y(Px \leftrightarrow Qy) \vdash \exists y \forall x(Px \supset Qy) \wedge \exists y \forall x(Qy \supset Px)$

11.8.2. Let B be a formula containing no free occurrences of x. Show that the formula

$$(\exists x(A(x) \wedge (B \supset C(x)))) \supset (\forall x(A(x) \supset \neg C(x)) \supset \neg B)$$

is logically valid in our predicate calculus. (The expression '$\phi(x)$' indicates that x occurs as a free variable in ϕ. That is, '$\phi(x)$' abbreviates '$\phi^{x/a}$'.)

Hint: reflect on the meaning of '$\phi^{x/a}$' and on the meaning of logical validity. Examine the tableau for the negation of the formula.

Solution. To verify the claim we build a tableau for the negation of our formula:

$$\mathbf{F}(\exists x(A^{x/a} \wedge (B \supset C^{x/a}))) \supset (\forall x(A^{x/a} \supset \neg C^{x/a}) \supset \neg B)$$

$$\mathbf{T}\exists x(A^{x/a} \wedge (B \supset C^{x/a}))$$
$$\mathbf{F}\forall x(A^{x/a} \supset \neg C^{x/a}) \supset B$$

$$\mathbf{T}\forall x(A^{x/a} \supset \neg C^{x/a})$$
$$\mathbf{F}\neg B$$

$$\mathbf{T}A^{x/a} \wedge (B \supset C^{x/a})$$

$$\mathbf{T}A^{x/a}$$
$$\mathbf{T}B \supset C^{x/a}$$

$$\mathbf{F}B \qquad \mathbf{T}C^{x/a}$$
$$\times$$

$$\mathbf{T}A^{x/a} \supset \neg C^{x/a}$$

$$\mathbf{F}A^{x/a} \qquad \mathbf{T}\neg C^{x/a}$$
$$\times \qquad \times$$