Metaphysics // Fall 2019

Handout 2

Truth by convention I: Quine

Note: Page references are to the separate version of the paper on Moodle and not the one in our Reader. There are minor discrepancies between the two versions.

The problem. Having mentioned the increasing role of definitions and conventions in physics, Quine turns to the view that mathematics and logic are 'purely analytic or conventional'. He promises to challenge the meaning of this claim. 250

Definition as a notational convention. Often we speak of definitions in the cases where nothing more is required than a mere notational change. An example would be a definition of tangent in terms of sine and cosine. Such notational innovations can generate logical truths, but only in a very special sense.

Example 1. We define, in the above sense, $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Then we have the following derivation: 251

(i) $\frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{\cos(x)}$ (law of identity)
(ii) $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (conventional abbreviation, aka definition)
(iii) $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (from above, logical truth)

The last line is obtained by replacing the left-hand side expression with its abbreviation. It is a logical truth, because the law of identity is. It is, as Quine says, not a proper derivation: the inference based on abbreviation is not justified by any logical rule. We may label it as an abbreviation of logical truth.

Quine stealthily turns to the question whether mathematics is conventional and derivable from logic. 252
This would mean, in present terms, whether mathematical claims are, in a similar fashion, are definable (abbreviatable) as logical truths.

Definability. Every choice of notation, hence every notational abbreviation, is arbitrary (though not every notation is, therefore, successful or efficient). But the definition of tangent above is arbitrary also in the sense that there is no antecedent use of the expression 'tangent' and of the corresponding concept. The definition creates novel use *ex nihilo*. This is not generally the case. Sometimes we want 252
to define a new term in such a way as to conform to the existent use. Such definitions will have two criteria of adequacy: the formal criterion of eliminability and the criterion of conformity to existent use.

Remark 2. Definitions in this sense will have to be 'explications'. For this notion see Carnap, *Logical Foundations of Probability*.

Conformity in this sense means notational transformation *salva veritate*: every sentence true (false) in traditional use must come out as true (false) under the proposed definition.

Vacuous and essential. To discuss definability further, Quine introduces the distinction between 252
vacuous and essential occurrences of expressions in the given sentence. A term t occurs vacuously in S if for every expression e, if e belongs to the same syntactical category as t, replacing t with e would not change the truth value of S. An expression occurs essentially if at least one of those replacements alters the truth value of S (we ignore the special case mentioned by Quine in the brackets)

Once all possible replacements of the vacuous expressions are done, we can see the logical 'skeleton' of the sentence. So another way to describe essential occurrences is to say that they belong to the skeleton.

How definitions work. Next, Quine shows how in general to think of reducing mathematics (or 253
any other body of sentences) to logic. We are given the following inventory:

(i) S: a true sentence
(ii) E_1, E_2, E_3: expressions occurring vacuously in S
(iii) S_1, S_2, S_3: vacuous variants of S

(iv) α: a certain class of expressions (eventually meant to be the class of logical constants)
(v) F: an expression occurring essentially in S

Suppose we attempt a definition $d(\alpha, F)$ of F on the basis of expressions in α (this definition is itself a string of expressions). Then let S' and S'_i be variations of S and S_i where F was replaced with its definition (but only outside the expressions E_i; we can skip this proviso). Also, S'_i is obtained from S' by replacing E_i by an acceptable grammatical variant.

Now, since the definition of F is supposed to conform to the antecedent use, the sentences S, S_i, S', S'_i are all true (for every i). Therefore, S'_i is a vacuous variant of S' (this was obvious all along, but Quine insists on spelling out every step). So S' has the same terms as S, but differs only in having a definition $d(\alpha, F)$ instead of F itself.

Simplifying a little (since we skipped the above proviso), we now say that F does not occur essentially in S'; rather $d(\alpha, F)$ occurs essentially in its stead. And now we repeat the same procedure for every other expression that occurs essentially in the original sentence S. The output is a sentence S'' where the only expressions occurring essentially are logical constants from α.

Example 3. Consider the statement (1a). We identify in it expressions occurring vacuously ('dog') and essentially ('one', 'and'). Then we define (paraphrase) the essential expressions in terms of logical constants.

(1) a. S: Putting one dog and another dog together we get two dogs.
b. S: One dog and another dog equals two dogs.
c. S_i: One cat and one cat equals two cats.
d. Not: Two dogs and two dogs equals two dogs.
e. Not: One dog or one dog equals two dogs.
f. S'': $\exists x(Dx \,\&\, \exists y(Dy \,\&\, x \neq y)) \leftrightarrow \exists x \exists y(Dx \,\&\, Dy \,\&\, x \neq y)$

Conditional reduction of mathematics to logic. In (1f) we eliminated, to our satisfaction, mathematical expressions ('one', 'two', 'equals') in favour of logical ones. But if this notational 254
transformation fails, then at least the non-mathematical expressions may be contained in mathematical expressions only vacuously (as before, we endorse the vacuous/essential dichotomy, ignoring Quine's warnings).

Thus consider geometry. It may be that 'sphere' and 'includes' are primitive expressions, but the other geometrical expressions can be defined on the basis of these two plus logical constants.

This is the case with Huntington geometry (see the printout) where the postulates only include the 254
two expressions as non-logical ones. With appropriate stipulations (see the text), we get:

(2) Φ(sphere, includes) $\approx$ [Hunt(sphere, includes) $\Rightarrow$ Φ(sphere, includes)].

In (2) we celebrate the idea that mathematical truths are, when rightly understood, theorems in an axiomatic system. In other words, we are in effect now saying that a mathematical statement such as Φ(sphere, includes) is in reality (deep down) a conditional statement. Now such a conditional statement is, in Huntington's terminology, a piece of abstract geometry. It is a logical truth, in which 'sphere' and 'includes' occur vacuously (as observed, in a different way, by Huntington himself).

But even if we grant that the whole conditional is now renamed as a mathematical truth, there still 255:1
remains the original statement Φ which is true according to the 'traditional use' already mentioned earlier. Perhaps we can call such use 'non-mathematical geometry', or with Huntington, 'concrete geometry'. This antecedent use, this part of traditional mathematics, was not reduced to logic. We can show that certain other discourses (mythology, sociology, botany, as suggested by Huntington himself) are axiomatisable by replacing their statements with appropriate conditionals. But of course we would not wish to conclude that these other discourses are, for this reason, purely logical!

Literal reduction of mathematics to logic. To be able to reduce mathematics to logic in 255:2
an acceptable sense one has to define every mathematical expression that occurs in a mathematical sentence 'under a literal reading' to a combination of logical expressions. Since mathematical expressions are not expected to be vacuous, we should demand that *all* mathematical expressions are defined in the above sense.

Furthermore, because of the wide use of mathematics in different (really, all) areas of discourse, 255:3
we should also demand that the proposed reduction respects the traditional use. That is, when

the definitional transcriptions have been completed, the newly formed logical definitions can be substituted *salva veritate* for the defined mathematical terms everywhere.

Quine's claim now is that contextual definitions, though they may be faithful to the use of pure mathematics, might not at the same time stay faithful to the use elsewhere.

Consider the statement (3a). It can be paraphrased in Huntington's geometry into the statement (3b). But 'Hunt(α, includes)' is true also when α is the class of spheres of foot or more in diameter (i.e. when it is restricted to some arbitrary class of spheres). But a baseball does not belong in this class. So the antecedent of (3b) is true, whilst the consequent is false. Hence the whole statement (3b) is false. Since, as we assumed, the original statement (3a) *is* true, the statement (3b) does not satisfy the demand of preserving truth value in every area of discourse. 257:1

(3) a. The whole of a baseball, except for a thin peripheral layer, constitutes a sphere.
b. If α is any class and R any relation such that Hunt(α, R), then the whole of a baseball, except for a thin peripheral layer, constitutes a member of α.

The problem restated It seems, therefore, that the task of reducing mathematics to logic is a much more difficult task than we imagined. This, Quine assumes, has been achieved in Whitehead and Russell's *Principia Mathematica*. 257:2

Still, even if mathematical reduction has been successfully accomplished, we cannot conclude that mathematical statements are true by convention. To claim this we have to show that *logic* is true by convention. The next task is to examine whether this is so. 258

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