THE PHILOSOPHY OF SPACE & TIME

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CHAPTER I. SPACE

§ 1. THE AXIOM OF THE PARALLELS AND NON-EUCLIDEAN GEOMETRY

Historical background

In Euclid's work, the geometrical achievements of the ancients reached their final form: geometry was established as a closed and complete system. The basis of the system was given by the geometrical axioms[1], from which all theorems were derived. The great practical significance of this construction consisted in the fact that it endowed geometry with a certainty never previously attained by any other science. The small number of axioms forming the foundation of the system were so self-evident that their truth was accepted without reservation. The entire construction of geometry was carried through by a skillful combination of the axioms alone, without any addition of further assumptions; the reliability of the logical inferences used in the proofs was so great that the derived theorems, which were sometimes quite involved, could be regarded as certain as the axioms. Geometry thus became the prototype of a demonstrable science, the first instance of a scientific rigor which, since that time, has been the ideal of every science. In particular, the philosophers of all ages have regarded it as their highest aim to prove their conclusions "by the geometrical method."

Euclid's axiomatic construction was also important in another respect. The problem of demonstrability of a science was solved by Euclid in so far as he had reduced the science to a system of axioms. But now arose the epistemological question how to justify the truth of those first assumptions. If the certainty of the axioms was transferred

[1] Euclid distinguished between axioms, postulates and definitions. We may be allowed for our present purpose to include all these concepts under the name of axioms.

to the derived theorems by means of the system of logical concatenations, the problem of the truth of this involved construction was transferred, conversely, to the axioms. It is precisely the assertion of the truth of the axioms which epitomizes the problem of scientific knowledge, once the connection between axioms and theorems has been carried through. In other words: the *implicational* character of mathematical demonstrability was recognized, i.e., the undeniable fact that only the implication "if *a*, then *b*" is accessible to logical proof. The problem of the categorical assertion "*a* is true *b* is true", which is no longer tied to the "if", calls for an independent solution. The truth of the axioms, in fact, represents the intrinsic problem of every science. The axiomatic method has not been able to establish knowledge with absolute certainty; it could only reduce the question of such knowledge to a precise thesis and thus present it for philosophical discussion.

This effect of the axiomatic construction, however, was not recognized until long after Euclid's time. Precise epistemological formulations could not be expected from a naive epoch, in which philosophy was not yet based upon well-developed special sciences, and thinkers concerned themselves with cruder things than the truth of simple and apparently self-evident axioms. Unless one was a skeptic, one was content with the fact that certain assumptions had to be believed axiomatically; analytical philosophy has learned mainly through Kant's critical philosophy to discover genuine problems in questions previously utilized only by skeptics in order to deny the possibility of knowledge. These questions became the central problems of epistemology. For two thousand years the criticism of the axiomatic construction has remained within the frame of mathematical questions, the elaboration of which, however, led to peculiar discoveries, and eventually called for a return to philosophical investigations.

The mathematical question concerned the reducibility of the axiomatic system, i.e., the problem whether Euclid's axioms represented ultimate propositions or whether there was a possibility of reducing them to still simpler and more self-evident statements. Since the individual axioms were quite different in character with respect to their immediacy, the question arose whether some of the more complicated axioms might be conceived as consequences of the simpler ones, i.e., whether they could be included among the theorems. In particular, the demonstrability of the axiom of the parallels was investigated. This axiom states that through a given point there is *one and only one* parallel to a given straight line (which does not go through the given

point), i.e., one straight line which lies in the same plane with the first one and does not intersect it. At first glance this axiom appears to be self-evident. There is, however, something unsatisfactory about it, because it contains a statement about infinity; the assertion that the two lines do not intersect within a finite distance transcends all possible experience. The demonstrability of this axiom would have enhanced the certainty of geometry to a great extent, and the history of mathematics tells us that excellent mathematicians from Proclus to Gauss have tried in vain to solve the problem.

A new turn was given to the question through the discovery that it was possible to do without the axiom of parallels altogether. Instead of proving its truth the opposite method was employed: it was demonstrated that this axiom could be dispensed with. Although the existence of several parallels to a given line through one point contradicts the human power of visualization, this assumption could be introduced as an axiom, and a consistent geometry could be developed in combination with Euclid's other axioms. This discovery was made almost simultaneously in the twenties of the last century by the Hungarian, Bolyai, and the Russian, Lobatschewsky; Gauss is said to have conceived the idea somewhat earlier without publishing it.

But what can we make of a geometry that assumes the opposite of the axiom of the parallels? In order to understand the possibility of a non-Euclidean geometry, it must be remembered that the axiomatic construction furnishes the proof of a statement in terms of logical derivations from the axioms alone. The drawing of a figure is only a means to assist visualization, but is never used as a factor in the proof; we know that a proof is also possible by the help of "badly-drawn" figures in which so-called congruent triangles have sides obviously different in length. It is not the immediate picture of the figure, but a concatenation of logical relations that compels us to accept the proof. This consideration holds equally well for non-Euclidean geometry; although the drawing looks like a "badly-drawn" figure, we can with its help discover whether the logical requirements have been satisfied, just as we can do in Euclidean geometry. This is why non-Euclidean geometry has been developed from its inception in an axiomatic construction; in contradistinction to Euclidean geometry where the theorems were known first and the axiomatic foundation was developed later, the axiomatic construction was the instrument of discovery in non-Euclidean geometry.

With this consideration, which was meant only to make non-Euclidean geometry plausible, we touch upon the problem of the *visualization of geometry.* Since this question will be treated at greater length in a later section, the remark about "badly-drawn" figures should be taken as a passing comment. What was intended was to stress the fact that the essence of a geometrical proof is contained in the logic of its derivations, not in the proportions of the figures. Non-Euclidean geometry is a logically constructible system—this was the first and most important result established by its inventors.

It is true that a strict proof was still missing. No contradictions were encountered—yet did this mean that none would be encountered in the future? This question constitutes the fundamental problem concerning an axiomatically constructed logical system. It is to be expected that non-Euclidean statements directly contradict those of Euclidean geometry; one must not be surprised if, for instance, the sum of the angles of a triangle is found to be smaller than two right angles. This contradiction follows necessarily from the reformulation of the axiom of the parallels. What is to be required is that the new geometrical system be self-consistent. The possibility can be imagined that a statement *a*, proved within the non-Euclidean axiomatic system, is not tenable in a later development, i.e., that the statement *not-a* as well as the statement *a* is provable in the axiomatic system. It was incumbent upon the early adherents of non-Euclidean geometry, therefore, to prove that such a contradiction could never happen.

The proof was furnished to a certain extent by Klein's[1] Euclidean model of non-Euclidean geometry. Klein succeeded in coordinating the concepts of Euclidean geometry, its points, straight lines, and planes, its concept of congruence, etc., to the corresponding concepts of non-Euclidean geometry, so that every statement of one geometry corresponds to a statement of the other. If in non-Euclidean geometry a statement *a* and also a statement *not-a* could be proved, the same would hold for the coordinated statements *a'* and *not-a'* of Euclidean geometry; a contradiction in non-Euclidean geometry would entail a corresponding contradiction in Euclidean geometry. The result was a proof of consistency, the first in the history of mathematics: it proceeds by reducing a new system of statements to an earlier one, the consistency of which is regarded as virtually certain.[2]

After these investigations by Klein the mathematical significance of

[1] For a more detailed presentation see § 11.

[2] Hilbert later proved the consistency of Euclidean geometry by a reduction to arithmetic. The consistency of arithmetic, which can no longer be proved by reduction, needs a separate proof; this most important problem, which has found an elaborate treatment by Hilbert and his school, is still under discussion.

non-Euclidean geometry was recognized.[1] Compared with the natural geometry of Euclid, that of Bolyai and Lobatschewsky appeared strange and artificial; but its mathematical legitimacy was beyond question. It turned out later that another kind of non-Euclidean geometry was possible. The axiom of the parallels in Euclidean geometry asserts that to a given straight line through a given point there exists exactly one parallel; apart from the device used by Bolyai and Lobatschewsky to deny this axiom by assuming the existence of several parallels, there was a third possibility, that of denying the existence of any parallel. However, in order to carry through this assumption consistently,[2] a certain change in a number of Euclid's other axioms referring to the infinity of a straight line was required. By the help of these changes it became possible to carry through this new type of non-Euclidean geometry.

The new status of axioms

As a result of these developments there exists not one geometry but a plurality of geometries. With this mathematical discovery, the epistemological problem of the axioms was given a new solution. If mathematics is not required to use certain systems of axioms, but is in a position to employ the axiom *not-a* as well as the axiom *a*, then the assertion *a* does not belong in mathematics, and mathematics is solely the science of implication, i.e., of relations of the form "if . . . then"; consequently, for geometry as a mathematical science, there is no problem concerning the truth of the axioms. This apparently unsolvable problem turns out to be a pseudo-problem. The axioms are not true or false, but arbitrary statements. It was soon discovered that the other axioms could be treated in the same way as the axiom of the parallels. "Non-Archimedian," "non-Pascalian," etc., geometries were constructed; a more detailed exposition will be found in § 14.

Geometrical truth

These considerations leave us with the problem into which discipline the question of the truth of the assertion *a* should be incorporated.

[1] Klein did not start his investigations with the avowed purpose of establishing a proof of consistency; the proof came about inadvertently, so to speak, as a result of the construction of the model carried out with purely mathematical intentions. L. Bieberbach has shown recently that the recognition of the significance of non-Euclidean geometry was the result of long years of struggle. *Berl. Akademieber.* 1925, phys.-math. Klasse, p. 381. See Bonola-Liebmann, *Nichteuklidische Geometrie*, Leipzig 1921 and Engel-Stäckel, *Theorie der Parallellinien von Euklid bis Gauss*, Leipzig 1895, for the earlier history of the axiom of the parallels.

[2] The axiom of the parallels is independent of the other axioms of Euclid only in so far as it asserts the existence of at most one parallel; that there exists at least one parallel can be demonstrated in terms of the other axioms. This fact is stated with masterful precision in Euclid's work.

Nobody can deny that we regard this statement as meaningful; common sense is convinced that real space, the space in which we live and move around, corresponds to the axioms of Euclid and that with respect to this space *a* is true, while *not-a* is false. The discussion of this statement leads away from mathematics; as a question about a property of the physical world, it is a *physical* question, not a *mathematical* one. This distinction, which grew out of the discovery of non-Euclidean geometry, has a fundamental significance: it divides the problem of space into two parts; the problem of mathematical space is recognized as different from the problem of physical space.

Mathematical geometry and physical geometry

It will be readily understood that the philosophical insight into the twofold nature of space became possible only after mathematics had made the step from Euclid's geometry to non-Euclidean geometries. Up to that time physics had assumed the axioms of geometry as the self-evident basis of its description of nature. If several kinds of geometries were regarded as mathematically equivalent, the question arose which of these geometries was applicable to physical reality; there is no necessity to single out Euclidean geometry for this purpose. Mathematics shows a variety of possible forms of relations among which physics selects the real one by means of observations and experiments. Mathematics, for instance, teaches how the planets would move if the force of attraction of the sun should decrease with the second or third or *n*th power of the distance; physics decides that the second power holds in the real world. With respect to geometry there had been a difference; only *one* kind of geometry had been developed and the problem of choice among geometries had not existed. After the discoveries of non-Euclidean geometries the duality of *physical* and *possible* space was recognized. Mathematics reveals the possible spaces; physics decides which among them corresponds to physical space. In contrast to all earlier conceptions, in particular to the philosophy of Kant, it becomes now a task of physics to determine the geometry of physical space, just as physics determines the shape of the earth or the motions of the planets, by means of observations and experiments.

Physical geometry is an empirical discipline

But what methods should physics employ in order to come to a decision? The answer to this question will at the same time supply an answer to the question why we are justified in speaking of a specific physical space. Before this problem can be investigated more closely, another aspect of geometry will have to be discussed. For physics the analytic treatment of geometry became even more fruitful than the axiomatic one.

§ 2. RIEMANNIAN GEOMETRY

This section can be consulted to clarify the technical ideas of the Einstein selection

Riemann's extension of the concept of space did not start from the axiom of the parallels, but centered around the concept of metric.

Riemann developed further a discovery by Gauss according to which the shape of a curved surface can be characterized by the geometry within the surface. Let us illustrate Gauss' idea as follows. We usually characterize the curvature of the surface of a sphere by its deviation from the plane; if we hold a plane against the sphere it touches only at one point; at all other points the distances between plane and sphere become larger and larger. This description characterizes the curvature of the surface of the sphere "from the outside"; the distances

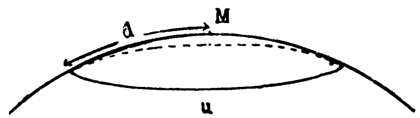

Fig. 1. Circumference and diameter of a circle on the surface of a sphere.

between the plane and the surface of the sphere lie outside the surface and the decision about the curvature has to make use of the third dimension, which alone establishes the difference between curved and straight. Is it possible to determine the curvature of the surface of the sphere without taking outside measurements? Is it meaningful to distinguish the curved surface from the plane within two dimensions? Gauss showed that such a distinction is indeed possible. If we were to pursue "practical geometry" on the sphere, by surveying, for instance, with small measuring rods, we should find out very soon that we were living on a curved surface. For the ratio of circumference u and diameter d of a circle we would obtain a number smaller than $\pi = 3.14...$ as is shown in Fig. 1. Since we stay on the surface all the time, we would not measure the "real diameter" which cuts through the inner part of the sphere, but the "curved diameter" which lies on the surface of the sphere and is longer. This diameter divided into the circumference results in a number smaller than π. Nevertheless, it is meaningful to call the point M "the center of the circle on the surface of the sphere" because it has the same distance from every point of the circle; that we find ourselves on a sphere is noticed by means of the deviation of the ratio from π. In this way we obtain a *geometry of a* *spherical surface* which is distinguished from the ordinary geometry by the fact that different metrical relations hold for this kind of geometry. In addition to the change in the ratio between circumference and diameter of a circle, an especially important feature is that the sum of the angles of a triangle on a sphere is greater than 180°.

It is remarkable that this generalization of plane geometry to surface geometry is identical with that generalization of geometry which originated from the analysis of the axiom of the parallels. The leading role which has been ascribed to the axiom of the parallels in the course of the development of geometrical axiomatics cannot be justified from a purely axiomatic point of view; the construction of non-Euclidean geometries could have been based equally well upon the elimination of other axioms. It was perhaps due to an intuitive feeling for theoretical fruitfulness that the criticism always centered around the axiom of the parallels. For in this way the axiomatic basis was created for that extension of geometry in which the metric appears as an independent variable.[1] Once the significance of the metric as the characteristic feature of the plane has been recognized from the viewpoint of Gauss' plane theory, it is easy to point out, conversely, its connection with the axiom of the parallels. The property of the straight line of being the shortest connection between two points can be transferred to curved surfaces, and leads to the concept of *straightest line*; on the surface of the sphere the great circles play the role of the shortest line of connection, and on this surface their significance is analogous to that of the straight lines on the plane. Yet while the great circles as "straight lines" share their most important property with those of the plane, they are distinct from the latter with respect to the axiom of the parallels: all great circles of the sphere intersect and therefore there are no parallels among these "straight lines". Here we encounter the second possibility of a denial (cf. § 1) of the axiom of the parallels which excludes the existence of parallels. If this idea is carried through, and all axioms are formulated on the understanding that by "straight lines" are meant the great circles of the sphere and by "plane" is meant the surface of the sphere, it turns out that this system of elements satisfies a system of axioms within two dimensions which is nearly identical in all of its statements with the axiomatic system of Euclidean geometry; *the only exception is the formulation of the axiom of the parallels.*[1] The geometry of the spherical surface can

[1] Cf. p. 148f about the connection of the axiom of the parallels with the metric.

be viewed as the realization of a two-dimensional non-Euclidean geometry: *the denial of the axiom of the parallels singles out that generalization of geometry which occurs in the transition from the plane to the curved surface.*

Once this result has been recognized for two-dimensional structures, a new kind of insight is gained into the corresponding problem of several dimensions by means of a combination of the two different points of departure. The axiomatic development of non-Euclidean geometry had already been achieved for three-dimensional structures and therefore constituted an extension of three-dimensional space analogous to the relation of the plane to the curved surface. Although Euclidean space contains curved surfaces, it does not embody the degree of logical generalization that characterizes the surfaces; it can realize only the Euclidean axiom of the parallels, not the axioms contradicting the latter. This fact suggests a concept of space which contains the plane Euclidean space as a special case, but includes all non-Euclidean spaces too. Such a concept of space in three dimensions is analogous to the concept of surface in two dimensions; it has the same relation to Euclidean space as a surface has to the plane.

On the basis of these ideas Riemann could give so generalized a definition to the concept of space that it includes not only Euclidean space but also Lobatschewsky's space as special cases. According to Riemann, space is merely a three-dimensional manifold; the question is left open which axiomatic systems will hold for it. Riemann showed that it is not necessary to develop an axiomatic system in order to find the different types of space; it is more convenient to use an analytic procedure analogous to the method developed by Gauss for the theory of surfaces. The geometry of space is established in terms of six functions, the *metrical coefficients of the line element*, which must be given [2] as a function of the coordinates; the manipulation of these functions replaces geometrical considerations, and all properties of geometry can be expressed analytically. This procedure can be

[1] It is evident, in considering the spherical surface, that two great circles will intersect in two points; hence, the denial of the axiom that two straight lines can intersect in only one point is involved. For if all of the axioms of Euclidean geometry except the parallel axiom are unchanged it is possible to prove there is at least one parallel. In the treatment of the spherical surface, however, we have seen that this theorem does not hold. This theorem depends upon the axiom that straight lines intersect in only one point; hence its denial removes the inconsistency.

[2] Cf. the more detailed presentation in § 39.

likened to the method in elementary analytic geometry which establishes an equivalence between a formula with two or three variables and a curve or a surface. The imagination is thus given conceptual support that carries it to new discoveries. In analogy to the auxiliary concept of the curvature of a surface, which is measured by the reciprocal product of the main radii of curvature, Riemann introduced the auxiliary concept of *curvature of space*, which is a much more complicated mathematical structure. Euclidean space, then, has a curvature of degree zero in analogy to the plane, which is a surface of zero curvature. Euclidean space occupies the middle ground between the spaces of positive and negative curvatures: it can be shown that this classification corresponds to the three possible forms of the axiom of the parallels. In the space of positive curvature *no* parallel to a given straight line exists; in the space of zero curvature *one* parallel exists; in the space of negative curvature *more than one* parallel exists. In general, the curvature of space may vary from point to point in a manner similar to the point to point variation in the curvature of a surface; but the spaces of *constant curvature* have a special significance. The space of constant negative curvature is that of Bolyai-Lobatschewsky; the space of constant zero curvature is the Euclidean space; the space of constant positive curvature is called spherical, because it is the three-dimensional analogue to the surface of the sphere. The analytical method of Riemann has led to the discovery of more types of space than the synthetic method of Bolyai and Lobatschewsky, which led only to certain spaces of constant curvature. Modern mathematics treats all these types of space on equal terms and develops and manipulates their properties as easily as those of Euclidean geometry.

§ 3. THE PROBLEM OF PHYSICAL GEOMETRY

How to determine geometrical truth?

Let us now return to the question asked at the end of § 1. The geometry of physical space had to be recognized as an empirical problem; it is the task of physics to single out the *actual* space, i.e., physical space, among the *possible* types of space. It can decide this question only by empirical means: but how should it proceed?

The method for this investigation is given by Riemann's mathematical procedure: the decision must be brought about by *practical*

measurements in space. In a similar way as the inhabitants of a spherical surface can find out its spherical character by taking measurements, just as we humans found out about the spherical shape of our earth which we cannot view from the outside, it must be possible to find out, by means of measurements, the geometry of the space in which we live. *There is a geodetic method of measuring space analogous to the method of measuring the surface of the earth.* However, it would be rash to make this assertion without further qualification. For a clearer understanding of the problem we must once more return to the example of the plane.

We can find out the geometry of physical space by measurement

Compare Poincare's heated sphere

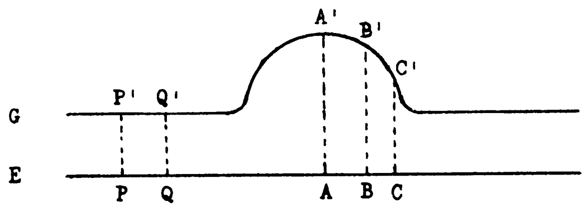

Fig. 2. Projection of a non-Euclidean geometry on a plane.

The setup of the thought-experiment

Let us imagine (Fig. 2) a big hemisphere made of glass which merges gradually into a huge glass plane; it looks like a surface G consisting of a plane with a hump. Human beings climbing around on this surface would be able to determine its shape by geometrical measurements. They would very soon know that their surface is plane in the outer domains but that it has a hemispherical hump in the middle; they would arrive at this knowledge by noting the differences between their measurements and two-dimensional Euclidean geometry.

An opaque plane E is located below the surface G parallel to its plane part. Vertical light rays strike it from above, casting shadows of all objects on the glass surface upon the plane. Every measuring rod which the G-people are using throws a shadow upon the plane; we would say that these shadows suffer deformations in the middle area. The G-people would measure the distances $A'B'$ and $B'C'$ as equal in length, but the corresponding distances of their shadows AB and BC would be called unequal.

The behaviour of measuring rods on E

Let us assume that the plane E is also inhabited by human beings and let us add another strange assumption. On the plane a mysterious force varies the length of all measuring rods moved about in that plane, so that they are always equal in length to the corresponding shadows projected from the surface G. Not only the measuring rods, however, but all objects, such as all the other measuring instruments and the bodies of the people themselves, are affected in the same way; these people, therefore, cannot directly perceive this change. What kind of measurements would the E-people obtain? In the outer areas of the plane nothing would be changed, since the distance $P'Q'$ would be projected in equal length on PQ. But the middle area which lies below the glass hemisphere would not furnish the usual measurements. Obviously the same results would be obtained as those found in the middle region by the G-people. Assume that the two worlds do not know anything about each other, and that there is no outside observer able to look at the surface E—what would the E-people assert about the shape of their surface?

Measurement is crucial in the adoption of geometry

They would certainly say the same as the G-people, i.e., that they live on a plane having a hump in the middle. They would not notice the deformation of their measuring rods. But why would they not notice this deformation?

We can easily imagine it to be caused by a physical factor, for instance by a source of heat under the plane E, the effects of which are concentrated in the middle area. It expands the measuring rods so that they become too long when they approach A. Geometrical relations similar to those we assumed would be realized; the distances CB and BA would be covered by the same measuring rod and heat would be the mysterious force we imagined.

Could deformation be discovered?

But could the E-people discover this force? Before we answer this question we have to formulate it more precisely. If the E-people knew that their surface is really a plane, they could, of course, notice the force by the discrepancy between their observed geometry and Euclidean plane geometry. The question, therefore, should read: how can the effect of the force be discovered if the nature of the geometry is not known? Or better still: how can the force be detected if the nature of the geometry may not be used as an indicator?

Differential and universal forces

If heat were the affecting force, ***direct*** indications of its presence could be found which would not make use of geometry as an ***indirect*** method. The E-people would discover the heat by means of their sense of temperature. But they would be able to demonstrate the heat expansion independently of this sensation, due to the fact that heat affects different materials in different ways. Thus the E-people would obtain one geometry when using copper measuring rods and another when using wooden measuring rods. In this way they would

notice the existence of a *force*. Indeed, direct evidence for the presence of heat is based on the fact that it affects different materials in *different* ways. The fact that the difference in temperature at the points A and C is demonstrable by the help of a thermometer is based on this phenomenon; if the mercury did not expand more than the glass tube and the scale of the thermometer, the instrument would show the same reading at all temperatures. Even the physiological effect of heat upon the human body depends upon differences in the reactions of different nerve endings to heat stimuli.

Heat as a force can thus be demonstrated directly. The forces, however, which we introduced in our example, cannot be demonstrated directly. They have two properties:

(a) They affect all materials in the same way.
(b) There are no insulating walls.

We have discussed the first property, but the second one is also necessary if the deformation is to be taken as a purely metrical one; it will be presented at greater length in § 5. For the sake of completeness the definition of the insulating wall may be added here: it is a covering made of any kind of material which does not act upon the enclosed object with forces having property ***a***. Let us call the forces which have the properties ***a*** and ***b*** ***universal forces***; all other forces are called ***differential forces***. Then it can be said that only differential forces, but not universal forces, are directly demonstrable.

What is the "real" geometry?

After these considerations, what can be stated about the shape of the surfaces E and G? G has been described as a surface with a hump and E as a plane which appears to have a hump. By what right do we make this assertion? The measuring results are the same on both surfaces. If we restrict ourselves to these results, we may just as well say that G is the surface with the "illusion" of the hump and E the surface with the "real" hump. Or perhaps both surfaces have a hump. In our example we assumed from the beginning that E was a plane and G a surface with a hump. By what right do we distinguish between E and G? Does E differ in any respect from G?

These considerations raise a strange question. We began by asking for the actual geometry of a real surface. We end with the question: Is it meaningful to assert geometrical differences with respect to real surfaces? This peculiar indeterminacy of the problem of physical geometry is an indication that something was omitted in the formulation of the problem. We forgot that a unique answer can only be found if the question has been stated exhaustively. Evidently some assumption is missing. Since the determination of geometry depends on the question whether or not two distances are really equal in length (the distances AB and BC in Fig. 2), we have to know beforehand what it means to say that two distances are "really equal." Is *really equal* a meaningful concept? We have seen that it is impossible to settle this question if we admit universal forces. Is it, then, permissible to ask the question?

Let us therefore inquire into the epistemological assumptions of measurement. For this purpose an indispensable concept, which has so far been overlooked by philosophy, must be introduced. The concept of a *coordinative definition* is essential for the solution of our problem.

§ 4. COORDINATIVE DEFINITIONS

Two kinds of definitions

Compare Einstein on practical geometry

Coordination is not arbitrary

Defining usually means reducing a concept to other concepts. In physics, as in all other fields of inquiry, wide use is made of this procedure. There is a second kind of definition, however, which is also employed and which derives from the fact that physics, in contradistinction to mathematics, deals with real objects. Physical knowledge is characterized by the fact that concepts are not only defined by other concepts, but are also coordinated to real objects. This coordination cannot be replaced by an explanation of meanings, it simply states that *this concept* is coordinated to *this particular thing*. In general this coordination is not arbitrary. Since the concepts are interconnected by testable relations, the coordination may be verified as true or false, if the requirement of uniqueness is added, i.e., the rule that the same concept must always denote the same object. The method of physics consists in establishing the uniqueness of this coordination, as Schlick[1] has clearly shown. But certain preliminary coordinations must be determined before the method of coordination can be carried through any further; these first coordinations are therefore definitions which we shall call *coordinative definitions*. They are *arbitrary*, like all definitions; on their choice depends the conceptual system which develops with the progress of science.

Wherever metrical relations are to be established, the use of coordinative definitions is conspicuous. If a distance is to be measured,

[1] M. Schlick, *Allgemeine Erkenntnislehre*, Springer, Berlin 1918, Ziff. 10.

Definition of length

To define length we require a measuring rod

the unit of length has to be determined beforehand by definition. This definition is a coordinative definition. Here the duality of conceptual definition and coordinative definition can easily be seen. We can define only by means of other concepts what we mean by a unit; for instance: "A unit is a distance which, when transported along another distance, supplies the measure of this distance." But this statement does not say anything about the size of the unit, which can only be established by reference to a physically given length such as the standard meter in Paris. The same consideration holds for other definitions of units. If the definition reads, for instance: "A meter is the forty-millionth part of the circumference of the earth," this circumference is the physical length to which the definition refers by means of the insertion of some further concepts. And if the wave-length of cadmium light is chosen as a unit, cadmium light is the physical phenomenon to which the definition is related. It will be noticed in this example that the method of coordinating a unit to a physical object may be very complicated. So far nobody has seen a wave-length; only certain phenomena have been observed which are theoretically related to it, such as the light and dark bands resulting from interference. In principle, a unit of length can be defined in terms of an observation that does not include any metrical relations, such as "that wave-length which occurs when light has a certain redness." In this case a sample of this red color would have to be kept in Paris in place of the standard meter. The characteristic feature of this method is the coordination of a concept to a physical object. These considerations explain the term "coordinative definition." If the definition is used for measurements, as in the case of the unit of length, it is a *metrical* coordinative definition.

Compare Bridgman

The meaning of "relativity"

The philosophical significance of the theory of relativity consists in the fact that it has demonstrated the necessity for metrical coordinative definitions in several places where empirical relations had previously been assumed. It is not always as obvious as in the case of the unit of length that a coordinative definition is required before any measurements can be made, and pseudo-problems arise if we look for truth where definitions are needed. The word "relativity" is intended to express the fact that the results of the measurements depend upon the choice of the coordinative definitions. It will be shown presently how this idea affects the solution of the problem of geometry.

After this solution of the problem of the unit of length, the next step leads to the comparison of two units of lengths at different locations.

If the measuring rod is laid down, its length is compared only to that part of a body, say a wall, which it covers at the moment. If two separate parts of the wall are to be compared, the measuring rod will have to be transported. It is assumed that the measuring rod does not change during the transport. It is fundamentally impossible, however, to detect such a change if it is produced by universal forces. Assume two measuring rods which are equal in length. They are transported by different paths to a distant place; there again they are aid down side by side and found equal in length. Does this procedure prove that they did not change on the way? Such an assumption would be incorrect. The only observable fact is that the two measuring rods are always equal in length at the place where they are compared to each other. But it is impossible to know whether on the way the two rods expand or contract. An expansion that affects all bodies in the same way is not observable because a direct comparison of measuring rods at different places is impossible.

Universal deformation of rods is undetectable

An optical comparison, for instance by measuring the angular perspective of each rod with a theodolite, cannot help either. The experiment makes use of light rays and the interpretation of the measurement of the lengths depends on assumptions about the propagation of light.

The problem does not concern a matter of *cognition* but of *definition.* There is no way of knowing whether a measuring rod retains its length when it is transported to another place; a statement of this kind can only be introduced by a definition. For this purpose a coordinative definition is to be used, because two physical objects distant from each other are *defined* as equal in length. It is not the *concept* equality of length which is to be defined, but a *real object* corresponding to it is to be pointed out. A physical structure is coordinated to the concept equality of length, just as the standard meter is coordinated to the concept unit of length.

Rods have the same length at different places by definition

This analysis reveals how definitions and empirical statements are interconnected. As explained above, it is an observational fact, formulated in an empirical statement, that two measuring rods which are shown to be equal in length by local comparison made at a certain space point will be found equal in length by local comparison at every other space point, whether they have been transported along the same or different paths. When we add to this empirical fact the definition that the rods shall be called equal in length when they are at *different places*, we do not make an inference from the observed fact; the addition

constitutes an independent convention. There is, however, a certain relation between the two. The physical fact makes the convention unique, i.e., independent of the path of transportation. The statement about the uniqueness of the convention is therefore empirically verifiable and not a matter of choice. One can say that the factual relations holding for a local comparison of rods, though they do not require the definition of congruence in terms of transported rods, make this definition admissible. Definitions that are not unique are inadmissible in a scientific system.

Factual conditions in our world allow for a single rod

This consideration can only mean that the factual relations may be used for the simple definition of congruence where any rigid measuring rod establishes the congruence. If the factual relations did not hold, a special definition of the unit of length would have to be given for every space point. Not only at Paris, but also at every other place a rod having the length of a "meter" would have to be displayed, and all these arbitrarily chosen rods would be called equal in length by definition. The requirement of uniformity would be satisfied by carrying around a measuring rod selected at random for the purpose of making copies and displaying these as the unit. If two of these copies were transported and compared locally, they would be different in length, but this fact would not "falsify" the definition. In such a world it would become very obvious that the concept of congruence is a definition; but we, in our simple world, are also permitted to choose a definition of congruence that does not correspond to the actual behavior of rigid rods. Thus we could arrange measuring rods, which in the ordinary sense are called equal in length, and, laying them end to end, call the second rod half as long as the first, the third one a third, etc. Such a definition would complicate all measurements, but epistemologically it is equivalent to the ordinary definition, which calls the rods equal in length. In this statement we make use of the fact that the definition of a unit at only one space point does not render general measurements possible. For the general case the definition of the unit has to be given in advance as a function of the place (and also of the time).[1] *It is again a matter of fact that our world admits of a simple definition of congruence because of the factual relations holding for the behavior of rigid rods; but this fact does not deprive the simple definition of its definitional character.*

Having rods at different place still requires definitions

The great significance of the realization that congruence is a matter of definition lies in the fact that by its help the epistemological problem

[1] Cf. § 39 and § 46.

Reformulation of the surfaces example

of geometry is solved. The determination of the geometry of a certain structure depends on the definition of congruence. In the example of the surface E the question arose whether or not the distances AB and BC are equal; in the first case the surface E will have the same geometrical form as the surface G, in the second case it will be a plane. The answer to this question can now be given in terms of the foregoing analysis: whether $AB = BC$ is not a matter of cognition but of definition. If in E the congruence of widely separated distances is defined in such a way that $AB = BC$, E will be a surface with a hump in the middle; if the definition reads differently, E will be a plane. *The geometrical form of a body is no absolute datum of experience, but depends on a preceding coordinative definition*; depending on the definition, the same structure may be called a plane, or a sphere, or a curved surface. Just as the measure of the height of a tower does not constitute an absolute number, but depends on the choice of the unit of length, or as the height of a mountain is only defined when the zero level above which the measurements are to be taken is indicated, geometrical shape is determined only after a preceding definition. This requirement holds for the three-dimensional domain in the same way as it does for the two-dimensional.

The case of three dimensions

While in the two-dimensional case the observed non-Euclidean geometry can be interpreted as the geometry of a curved surface in a Euclidean three-dimensional space, we arrive at a three-dimensional non-Euclidean geometry when we measure a three-dimensional structure. A simple consideration will clarify this point. Let us choose as our coordinative definition that of practical surveying, i.e., let us define rigid measuring rods as congruent, when transported. If under these conditions a large circle, say with a radius of 100 meters, is measured on the surface of the earth, a very exact measurement will furnish a number smaller than $\pi = 3.14...$ for the relation of circumference and diameter. This result is due to the curvature of the surface of the earth, which prevents us from measuring the real diameter going through the earth below the curved surface. In this case it would be possible to use the third dimension. If we add the third dimension, however, the situation becomes different. Imagine a large sphere made of tin which is supported on the inside by rigid iron beams; on the sphere and upon the iron scaffold people are climbing around who are measuring circumference and diameter at different points with the same measuring rods they used for the two-dimensional case. If this time the measuring result deviates from π, we must accept a three-dimensional non-Euclidean geometry which

can no longer be interpreted as the curvature of a surface in three-dimensional Euclidean space. We obtain this result because the coordinative definition of congruence was chosen as indicated above. A different geometry would have been obtained, if we had used, for instance, the coordinative definition of the earlier example, in which we called the measuring rod half its length after putting it down twice, a third its length after putting it down three times, etc. The question of the geometry of real space, therefore, cannot be answered before the coordinative definition is given which establishes the congruence for this space.

What is the true physical geometry?

We are now left with the problem: which coordinative definition should be used for physical space? Since we need a geometry, a decision has to be made for a definition of congruence. Although we must do so, we should never forget that we deal with an arbitrary decision that is neither true nor false. Thus the geometry of physical space is not an immediate result of experience, but depends on the choice of the coordinative definition.

Definitions are judged by their simplicity and conservatism

In this connection we shall look for the most adequate definition, i.e., one which has the advantage of logical simplicity and requires the least possible change in the results of science. The sciences have implicitly employed such a coordinative definition all the time, though not always consciously; the results based upon this definition will be developed further in our analysis. It can be assumed that the definition hitherto employed possesses certain practical advantages justifying its use. In the discussion about the definition of congruence by means of rigid rods, this coordinative definition has already been indicated. The investigation is not complete, however, because an exact definition of the *rigid body* is still missing.

§ 5. RIGID BODIES

Solid bodies do not change state, hence useful for defining congruence

Experience tells us that physical objects assume different states. Solid bodies have an advantage over liquid ones because they change their shape and size only very little when affected by outside forces. They seem, therefore, to be useful for the definition of congruence. However, if the result of the previous considerations is kept in mind, this relative stability is no ground on which to base a preference for solid bodies. As was explained, the form and size of an object depends on the coordinative definition of congruence; if the solid body is used for the coordinative definition, the statement that it does not change its

How to define rigidity?

shape must not be regarded as a cognitive statement. It can only be a definition: we define the shape of the solid body as unchangeable. But how can the solid body be defined? In other words, if the physical state of *being solid* were defined differently, under what conditions would the solid body be called *rigid*? If the conservation of shape is not permissible as a criterion, what criteria may be used?

Not by ostension

No rigids in nature

Can we salvage coordinative definitions?

The problem becomes more complicated because we cannot solve it by merely pointing to certain real objects. Although the standard meter in Paris was cited previously as the prototype of such a definition, this account was a somewhat schematic abstraction. Actually no object is the perfect realization of the rigid body of physics; it must be remembered that such an object may be influenced by many physical forces. Only after several corrections have been made, for example, for the influence of temperature and elasticity, is the resulting length of the object regarded as adequate for the coordinative definition of the comparison of lengths. The standard meter in Paris would not be accepted as the definition of the unit of length, if it were not protected from influences of temperature, etc., by being kept in a vault. If an earthquake should ever throw it out of this vault and deform its diameter, nobody would want to retain it as the prototype of the meter; everybody would agree that the standard meter would no longer be a meter. But what kind of definition is this, if the definition may some day be called false? Does the concept of coordinative definition become meaningless?

Practical constraints on coordinative definitions

The answer is: it does not become meaningless, but, as we shall see, its application is logically very complicated. The restrictions that affect the arbitrariness of the coordinative definition have two sources. One restriction lies in the demand that the obtained metric retain certain older physical results, especially those of the "physics of daily life." Nobody could object on logical grounds if the bent rod would be taken as the definition of the unit of length; but then we must accept the consequence that our house, our body, the whole world has become larger. Relative to the coordinative definition it has, indeed, become larger, but such an interpretation does not correspond to our habitual thinking. We prefer an interpretation of changes involving an individual thing on the one side and the rest of the world on the other side that confines the change to the small object. The theory of motion uses the same idea; the fly crawling around in the moving train is called "moving" relative to the train, and the train is called "moving" relative to the earth. Provided that we realize that such a description

No definition is better on logical grounds

cannot be justified on logical grounds, we can employ it without hesitation because it is more convenient; yet it must not be regarded as "more true" than any other description. We must not assume that a deformation of the standard meter by an earthquake is equivalent to a change in any absolute sense; actually it is only a change in the *difference* in size between the rod and the rest of the world. There is, of course, no objection to the use of such restrictions on coordinative definitions, because their only effect is an adaptation of the scientific definition to those of everyday life.

Better definitions deliver simpler physics

These restrictions are more numerous than might be anticipated offhand. Geometrical concepts abound in our daily life. We call the floor and ceiling plane, the corners of our rooms rectangular, a taut string straight. It is clear that these terms can only be definitions and have nothing to do with cognition, as one might at first believe. But by means of these definitions we have arrived at a very simple physics of everyday life. It would logically be permissible to define the taut string as curved, but then we would have to introduce a complicated field of force which pulls the string to the side and prevents it from adjusting itself to the shortest line in spite of the elastic tension, comparable to a stretched chain bending under the influence of gravity; such a convention would complicate physics unnecessarily. However, this is the only objection that can be raised against this description; the statement that a taut string is straight is not empirical but only a more convenient definition.

On the other hand, these restrictions do not constitute strict rules; they merely confine coordinative definitions to certain limits. Direct observation is inexact and we admit the possibility of small inaccuracies of observation. Scientifically speaking nobody will deny that the floor is a little curved, or that a tightened string sags slightly. Such a statement would mean that science does not really use the floor and the string but other physical objects as standards for its coordinative definition, and that, compared to these other things, small deviations occur. The physics of everyday life furnishes only limits for coordinative definitions; it does not intend to establish them strictly.

For everyday physics this strictness is not possible, and the task of scientific physics is therefore to give a strict formulation of the coordinative definition within these limits. This aim of precision is the reason for the important role played by correction factors and supplementary forces in the measurement of lengths. The principle according to which the strict definition is achieved must now be investigated more closely. What is the rigid body of physics? It must be defined strictly without the use of the concept of change in size.

Solidity and rigidity distinguished

For this purpose the concepts *rigid* and *solid* must be distinguished. Solid bodies are bodies having a certain physical state which can be defined ostensively; it differs from the liquid and gaseous state in a number of observable ways. The solid body can be defined without the use of the concept of change in size. Rigid bodies, however, are those bodies that constitute the physical part in the coordinative definition of congruence and that by definition do not change their size when transported. By the use of the concept *solid body* a definition of the concept *rigid body* can be given that does not employ congruence.

Rigidity defined by lack of differential effects

Definition: *Rigid bodies are solid bodies which are not affected by differential forces, or concerning which the influence of differential forces has been eliminated by corrections; universal forces are disregarded.*

Possibility of universal forces

This definition will be discussed presently. Let us first deal with the last clause. May we simply neglect universal forces? But we do not neglect them: we merely set the universal forces equal to zero by definition. Without such a rule the rigid body cannot be defined. Since there is no demonstrable difference produced by universal forces, the conception that the transported measuring rod is deformed by such forces can always be defended. No object is rigid relative to universal forces.

All physical forces are differential

This idea corresponds to the usual method of physics. All forces occurring in physics are differential forces in the sense of our definition. The terms "physical forces" and "differential forces" will therefore be used interchangeably in the following sections.

Rephrasing the definition

We must still discuss the first part of the definition of the rigid body. Again we shall use the method of the physicist. However, we shall avoid the vicious circle of defining the absence of exterior forces by an absence of change of shape. Since universal forces were eliminated by definition and exterior forces are always demonstrable by differential effects, the conservation of shape is defined inversely through the lack of exterior forces.

In nature, there are only approximately solid bodies

This rule needs an addition. It is not possible, even by computations, to eliminate exterior forces completely; small effects evade experimental observation and the definition supplies an ideal limit that can only be approximated. The method of approximation must therefore be discussed. Solid bodies possess considerable interior forces or tensions. According to the usual conception, these forces account for resistance against change of shape; but conversely, in our episte-

mological construction we can base the definition of negligible change of shape upon the occurrence of these interior forces and tensions. *Change of shape is called small if the exterior forces are small relative to the interior forces.* The more nearly this condition is realized, the more rigid is the body; but only at the unattainable limit where the exterior forces disappear relative to the interior forces would the rigid body be realized in the strict sense.

Rigid body represents a closed system

The definition of the rigid body depends on the definition of a *closed* system. Here lies the difficulty of the problem. Two critical points have been evaded by our definition. In the first place, a closed system can never be strictly realized; therefore, a transition to a limit must be given that permits us to call a system "closed to a certain degree of exactness." This transition to a limit is obtained through the relation between interior and exterior forces, which can be made very small by means of technical manipulation. Without the consideration of interior forces, however, the concept of a closed system could not be determined, because there is always a certain connection with the environment, and it is necessary to name the other magnitudes relative to which the exterior forces are small. It is, therefore, a necessary condition for a closed system to contain interior forces, and even in the transition to infinitesimal closed systems, exterior forces must vanish in a higher order than the interior ones. The second difficulty in the definition of closed systems lies in the possible existence of forces not demonstrable by differential measurements because they affect all indicators in the same way. Physical forces in the sense of our definition can be excluded by adequate protection; but if there exist forces which penetrate all insulating walls (property *b*, p. 13) there are no closed systems. As universal forces they were set equal to zero by definition and as such eliminated. Without such a rule a closed system cannot be defined.

Closed system contains interior forces

The existence of closed systems is postulated

Physics depends on the notion of rigid body

This definition of the rigid body is not explicitly given in the literature of physics, but it is that definition on which the whole system of physics is based. With a different definition physical laws would generally change; this follows from the fact that in the dimensions of the fundamental physical magnitudes, such as force and energy, the concept of length occurs; thus the values of these magnitudes depend on the definition of congruence. It must not be argued, however, that conversely the "truth" of our definition of congruence can be inferred from the truth of physical laws. The truth of the physical laws can only be asserted under the assumption of a definition of

Various concrete objects can be used as measuring rods, some will be more convenient than other

congruence; *the laws are true relative to the definition of congruence* by means of rigid bodies. The following example will illustrate this point: if a rubber band were used as the definition of congruence without any indication of its state of tension, the energy of closed systems would in general not be constant, since the measure of the energy would vary as a function of the rubber band. The kinetic energy would change, for instance, because the velocity of the body under consideration would vary with the changes in the rubber band. The law of conservation of energy would be replaced by a law stating the dependence of the energy of closed systems on the state of the rubber band. But this law would be just as true as the law of the conservation of energy. The disadvantage would consist only in the fact that the biography of the rubber band would have to be included in all physical laws. It is one of the most important facts of natural science that it is possible to establish physical laws free from such complications; the significance of the rigid body is based on it.

§ 6. THE DISTINCTION BETWEEN UNIVERSAL AND DIFFERENTIAL FORCES

Heat will be used as an example for the distinction between U- and D-forces

Our definition of the rigid body is based mainly upon the distinction between universal and differential forces. When we used heat as a differential force in our example above, we could show that a direct proof of physical forces is possible because of the difference of their effects on different materials. This idea must be elaborated further. The thermometer works because mercury and glass do not have the same coefficient of expansion. But can differences in temperature be demonstrated only by differences between the reactions of various materials to heat?

When we recall how the coefficient of expansion of a rod is measured

Device for observing the effects of D-forces

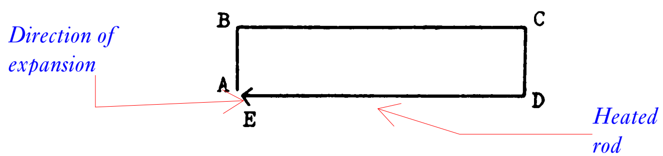

Fig. 3. Sketch of an apparatus for the measurement of heat expansion.

in practice, another possibility suggests itself. For this measurement a device is used as shown in Fig. 3. The distance ED corresponds to the rod to be measured. The end D is pressed firmly against the

support; the end E can move freely. Before the rod is heated, its length is equal to the distance AD. The heat is applied only to ED, while BC is kept at its initial temperature; thus the interval AD remains constant, while ED changes its length. E will move to the left beyond A. The influence of the heat is observable because A and E no longer coincide. This effect is observable, even if the whole apparatus consists of the same material. Imagine a copper wire bent in the rectangular shape of Fig. 3; the two ends of the wire meet in A and E. Such a device would be a "thermometer", because it would be possible to observe a change in temperature by the disappearance of the coincidence between A and E. Here the force is measured by an indicator made of only one material.

Copper wire as a thermometer

Such a device can serve quite generally to demonstrate the presence of forces; the indicator of the force will always react when the field of force is not homogeneous, i.e., if it affects the different parts of the wire in different ways. The field of force may fill the space continuously; if a measurement is to be taken in the field of heat, a complete insulation of the rod DE from the support, i.e., a discontinuity of the field of temperature, is not necessary for the qualitative demonstration of the expansion.

A more sophisticated device

The indicator can have yet another form that makes its operation even more obvious. Imagine a circle made of wire with a diameter

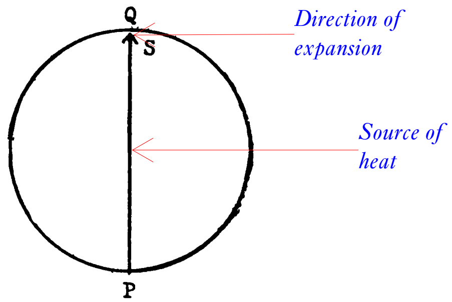

Fig. 4. Sketch of an indicator for the geometrical curvature.

of the same material (Fig. 4). At P this diameter is fastened to the circle, at Q the point S touches the ring, so that there is a coincidence between Q and S. Such an apparatus will also demonstrate the existence of higher temperature in the middle of the circle, for then Q

Meauring curvature of surface

will not coincide with S. The device can be used for other purposes, too. If it were moved along an egg-shaped surface so that the wire were everywhere in contact with the surface, Q would no longer coincide with S. The indicator points out directly the curvature of the surface by comparing circumference and diameter of a circle. Applied to surfaces of variable curvature, such as an egg-shaped surface, the indicator will register the curvature.

Geometrical change indicates a heat field if materials have different expansion coefficients

Here we have an indicator of geometrical relations, and we notice that evidence of a field of heat is furnished by a geometrical method. From the change of the geometry we infer the presence of the field of heat. We did not exclude the possibility of this inference; we must, however, analyze the question why, in this case, we go beyond the observation of the geometry, and infer a deforming force. Again we answer that the different reactions of the different kinds of material lead to this inference. In a field of heat the points Q and S on a copper indicator would be shifted in a different way than on an indicator made of iron wire; on an egg-shaped surface both would show the same differences. Thus the only distinguishing characteristic of a field of heat is the fact that it causes different effects on different materials.

If coefficients are the same, then the same change is evidence of space curvature

But we could very well imagine that the coefficients of heat expansion of all materials might be equal—then no difference would exist between a field of heat and the geometry of space. It would be permissible to say that in the neighborhood of a warm body the geometry is changed just as (according to Einstein) space is curved in the neighborhood of a large mass. Nothing could prevent us from carrying through this conception consistently. We do not adopt this procedure because we would then obtain a special geometry for copper, another one for iron, etc.; we avoid these complications by means of the definition of the rigid body.

Although we introduced the differential effects on different materials as an indication of physical forces, a test for forces is not necessarily bound to this difference. A field of force can be demonstrated by the help of one material alone, if the device is large enough to include inhomogeneities of the field. This method, however, will always provide indirect evidence only, because the observed change could equally well be interpreted as a change of geometry. That the change is interpreted as being due to a force can only be based on the unequal effect of the force on different materials. This criterion tells us what should be interpreted as physical deformation and what as geometry of space. The geometry of space, too, can be demonstrated objectively since its

physical effects are observable. The distinction between universal and differential forces merely classifies the phenomena as belonging in geometry or in physics.

A remark may be added concerning the treatment these questions receive in the literature. The forces which we called universal are often characterized as forces *preserving coincidences*; all objects are assumed to be deformed in such a way that the spatial relations of adjacent bodies remain unchanged. In this context belongs the assumption that overnight all things enlarge to the same extent, or that the size of transported objects is uniformly affected by their position. Helmholtz' parable of the spherical mirror comparing the world outside and inside the mirror is also of this kind; [1] if our world were to be so distorted as to correspond to the geometrical relations of the mirror images, we would not notice it, because all coincidences would be preserved. It has been correctly said that such forces are not demonstrable, and it has been correctly inferred that they have to be set equal to zero by definition if the question concerning the structure of space is to be meaningful. It follows from the foregoing considerations that this is a *necessary* but not a *sufficient* condition. Forces *destroying coincidences* must also be set equal to zero, if they satisfy the properties of the universal forces mentioned on p. 13; only then is the problem of geometry uniquely determined. Our concept of universal force is thus more general and contains the concept of the coincidence-preserving force as a special case. It should not be said, therefore, that universal forces are not demonstrable; this holds only for forces which preserve coincidences. Fig. 4, however, is an example of an indicator showing universal forces which destroy coincidences (in this case the coincidence QS).

Universal forces might not preserve coincidences

We can define such forces as equal to zero because a force is no absolute datum. When does a force *exist*? By force we understand something which is responsible for a *geometrical change*. If a measuring rod is shorter at one point than at another, we interpret this contraction as the effect of a force. The existence of a force is therefore dependent on the coordinative definition of geometry. If we say: actually a geometry G applies, but we measure a geometry G', we define at the same time a force F which causes the difference between G and G'. The geometry G constitutes the zero point for the magnitude of a force. If we find that there result several geometries G' according

Forces cause geometrical change

Space geometry is fixed by setting forces to 0

[1] H. v. Helmholtz, *Schriften zur Erkenntnistheorie*, ed. by Hertz and Schlick, Springer, Berlin 1921, p. 19.

How to distinguish D- and U-forces

as the material of the measuring instrument varies, F is a differential force; in this case we gauge the effect of F upon the different materials in such a way that all G' can be reduced to a common G. If we find, however, that there is only one G' for all materials, F is a universal force. In this case we can renounce the distinction between G and G', i.e., we can identify the zero point with G', thus setting F equal to zero. This is the result that our definition of the rigid body achieves.

§ 7. TECHNICAL IMPOSSIBILITY AND LOGICAL IMPOSSIBILITY

The problem: technical limitations and objective indeterminacy

In the following section a criticism will be discussed which has been made against our theory of coordinative definitions. It has been objected that we base the arbitrariness in the choice of the definition on the impossibility of making measurements. Although it is admitted that certain differences cannot be *verified* by measurement, we should not infer from this fact that they do not *exist*. If we had no means of discovering the shape of surface E in Fig. 2 (p. 11) it would still be meaningful to ask what shape the surface has; although the possibility of making measurements is dependent on our human abilities, the objective fact is independent of them. Thus we are accused of having confused *subjective inability* with *objective indeterminacy*.

Example of technical limitation

There are, indeed, many cases where physics is unable to make measurements. Does this mean that the magnitude to be measured does not exist? It is impossible, for instance, to determine exactly the number of molecules in a cubic centimeter of air; we can say with a high degree of certainty that we shall never succeed in counting every individual molecule. But can we infer that this number does not exist? On the contrary, we must say that there will always be an integer which denotes this quantity exactly. The mistake of the theory of relativity is supposed to consist in the fact that it confuses the *impossibility of making measurements* with *objective indeterminacy*.

The logical impossibility of measuring the measuring rod

Whoever makes this objection overlooks an important distinction. There is an impossibility of making measurements which is due to the limitation of our technical means; I shall call it *technical impossibility*. In addition, there is a *logical impossibility* of measuring. Even if we had a perfect experimental technique, we should not be able to avoid this logical impossibility. It is logically impossible to determine whether the standard meter in Paris is really a meter. The highest refinement of our geodetic instruments does not teach us anything

"The Paris rod is one meter long", "The rod here has the same length as the rod there" are meaningless

about this problem, because the meter cannot be defined in absolute terms. This is the reason why the measuring rod in Paris is called the definition of a meter. It is arbitrarily defined as the unit, and the question whether it really represents this unit has lost its meaning. The same considerations hold for a comparison of units at distant places. Here we are not dealing with technical limitations, but with a logical impossibility. The impossibility of a determination of the shape of a surface, if universal forces are admitted, is not due to a deficiency of our instruments, *but is the consequence of an unprecise question.*

"This surface has a shape S" is meaningless if universal forces are admitted

The question concerning the shape of the surface has no precise formulation, unless it is preceded by a coordinative definition of congruence. What is to be understood by "the shape of a real surface"? Whatever experiments and measurements I make, they will never furnish a unique indication of the shape of the surface. If universal forces are admitted, the measurements may be interpreted in such a way that many different shapes of surfaces are compatible with the same observations.

All these statements are meaningful upon fixing a coordinative definition

There is one definition which closes the logical gap and tells us which interpretations of our observations must be eliminated: this task is performed by the coordinative definition. It gives a precise meaning to the question of the shape of the real surface and makes a unique answer possible, just as a question about length has a unique meaning only when the unit of measurement is given. It is not a technical failure that prevents us from determining the shape of a surface without a coordinative definition of congruence, but a logical impossibility that has nothing to do with the limitations of human abilities.

The situation will be further clarified if we compare the last example with the case of the indeterminacy of the number of molecules in a given cubic centimeter of air. This number is precisely defined and it is only due to human imperfection that we cannot determine it exactly. But in this case an approximation is possible which will increase with increasing perfection of our technical instruments.

Logical impossibility is not due to approximations

When we are faced with a logical impossibility there are no approximations. We cannot decide approximately whether the surface E of Fig. 2 (p. 11) is a plane, or a surface with a hemispherical hump in the middle; there is no defined limit which the measurement could approach. Furthermore, once the coordinative definition is given, the technical impossibility of an exact measurement remains. Even our definition of the rigid body does not permit a strict determination of the structure of space; all our measurements will still contain some degree of inexactness which a progressive technique will gradually reduce but never overcome.

§ 8. THE RELATIVITY OF GEOMETRY

Physical geometry is empirical, depends on coordinative definitions

With regard to the problem of geometry we have come to realize that the question which geometry holds for physical space must be decided by measurements, i.e., empirically. Furthermore, this decision is dependent on the assumption of an arbitrary coordinative definition of the comparison of length. Against this conception arguments have been set forth which endeavor to retain Euclidean geometry for physical space under any circumstances and thus give it a preference among all other geometries. On the basis of our results we can discuss these arguments; our analysis will lead to the relativity of geometry.

Objection: measurement assumes E-geometry

One of the arguments maintains it is a mistake to believe that the choice of the coordinative definition is a matter left to our discretion. The measurements of geometry as carried through in practice presuppose quite complicated measuring instruments such as the theodolite; therefore these measurements cannot be evaluated without a theory of the measuring instruments. The theory of the measuring instruments, however, presupposes the validity of Euclidean geometry and it constitutes a contradiction to infer a non-Euclidean geometry from the results.

Reply: we have a choice, either to have E and U-forces, or to have non-E and no U-forces

This objection can be met in the following way. Our conception permits us to start with the assumption that Euclidean geometry holds for physical space. Under certain conditions, however, we obtain the result that there exists a universal force F that deforms all measuring instruments in the same way. However, we can invert the interpretation: we can set F equal to zero by definition and correct in turn the theory of our measuring instruments. We are able to proceed in this manner because a transformation of all measurements from one geometry into another is possible and involves no difficulties. It is correct to say that all measurements must be preceded by a definition; we expressed this fact by the indispensability of the coordinative definition. The mistake of the objection consists in the belief that this definition cannot be changed afterwards. Just as we can measure the temperature with a Fahrenheit thermometer and then convert the results into Celsius, measurements can be started under the assumption of Euclidean geometry and later converted into non-Euclidean measurements. There is no logical objection to this procedure.

We can adjust measuring tools after measurement...

In practice the method is much simpler. It turns out that the

non-Euclidean geometry obtained under our coordinative definition of the rigid body deviates quantitatively only very little from Euclidean geometry when small areas are concerned. In this connection "small area" means "on the order of the size of the earth"; deviations from Euclidean geometry can be noticed only in astronomic dimensions. In practice, therefore, it is not necessary to correct the theory of the measuring instruments afterwards, because these corrections lie within the errors of observation. The following method of inference is permissible: we can prove by the assumption that Euclidean geometry holds for small areas that in astronomic dimensions a non-Euclidean geometry holds which merges infinitesimally into Euclidean geometry. No logical objection can be advanced against this method, which is characteristic of the train of thought in modern physics. It is carried through in practice for astronomic measurements designed to confirm Einstein's theory of gravitation.

...but in practice this is not required

The objection is connected with the *a priori* theory of space that goes back to Kant and today is represented in various forms. Not only Kantians and Neo-Kantians attempt to maintain the *a priori* character of geometry: the tendency is also pronounced in philosophical schools which in other respects are not Kantian. It is not my intention to give a critical analysis of Kant's philosophy in the present book. In the course of the discussion of the theory of relativity, it has become evident that the philosophy of Kant has been subject to so many interpretations by his disciples that it can no longer serve as a sharply defined basis for present day epistemological analysis. Such an analysis would clarify less the *epistemological* question of the structure of space than the *historical* question of the meaning and content of Kant's system. The author has presented his own views on this problem in another publication;[1] the present investigation is aimed at philosophical clarification and will not concern itself with historical questions. Therefore, I shall select only those arguments of Kant's theory of space, the refutation of which will further our understanding of the problem. Although in my opinion the essential part of Kant's theory will thereby be covered, I do not claim a historically complete evaluation of it in this book.

The ideas expressed in the preceding considerations attempted to establish Euclidean geometry as *epistemologically a priori*; we found that this *a priori* cannot be maintained and that Euclidean geometry

[1] H. Reichenbach, *Relativitätstheorie und Erkenntnis a priori*, Springer, Berlin, 1920.

Is the visual space Euclidean?

is not an indispensable presupposition of knowledge. We turn now to the idea of the *visual a priori*; this Kantian doctrine bases the preference for Euclidean geometry upon the existence of a certain manner in which we visualize space.

The theory contends that an innate property of the human mind, the ability of visualization, demands that we adhere to Euclidean geometry. In the same way as a certain self-evidence compels us to believe the laws of arithmetic, a visual self-evidence compels us to believe in the validity of Euclidean geometry. It can be shown that this self-evidence is not based on logical grounds. Since mathematics furnishes a proof that the construction of non-Euclidean geometries does not lead to contradictions, no *logical* self-evidence can be claimed for Euclidean geometry. This is the reason why the self-evidence of Euclidean geometry has sometimes been derived, in Kantian fashion, from the human ability of visualization conceived as a source of knowledge.

Euclidean axioms are visually self-evident

Everybody has a more or less clear notion of what is understood by visualization. If we draw two points on a piece of paper, connect them by a straight line and add a curved connecting line, we "see" that the straight line is shorter than the curved line. We even claim to be certain that the straight line is shorter than any other line connecting the two points. We say this without being able to prove it by measurements, because it is impossible for us to draw and measure all the lines. The power of imagination compelling us to make this assertion is called the ability of *visualization*. Similarly, the Euclidean axiom of the parallels seems to be visually necessary. It remains for us to investigate this human quality and its significance for the problem of space.

The analysis will be carried through in two steps. Let us first assume it is correct to say that a special ability of visualization exists, and that Euclidean geometry is distinguished from all other geometries by the fact that it can easily be visualized. The question arises: what consequences does this assumption have for physical space? Only after this question has been answered can the assumption itself be tested. The second step of our analysis will therefore consist in the inquiry whether a special ability of visualization exists (§ 9–§ 11).

What are the implications for physical geometry?

Let us turn to the first question, which has to be reformulated in order to relate it clearly to the epistemological problem.

Mathematics proves that every geometry of the Riemannian kind can be mapped upon another one of the same kind. In the language of physics this means the following:

Physical geometry is indeterminate in the presence of unspecified U-forces

Theorem θ: "Given a geometry G' to which the measuring instruments conform, we can imagine a universal force F which affects the instruments in such a way that the actual geometry is an arbitrary geometry G, while the observed deviation from G is due to a universal deformation of the measuring instruments."[1]

No epistemological objection can be made against the correctness of theorem θ. Is the visual *a priori* compatible with it?

Offhand we must say yes. Since the Euclidean geometry G_0 belongs to the geometries of the Riemannian kind, it follows from theorem θ that it is always possible to carry through the visually preferred geometry for physical space. Thus we have proved that we can always satisfy the requirement of visualization.

The question of true geometry is meaningless

But something more is proved by theorem θ which does not fit very well into the theory of the visual *a priori*. The theorem asserts that Euclidean geometry is not preferable on epistemological grounds. Theorem θ shows all geometries to be equivalent; it formulates the *principle of the relativity of geometry*. It follows that it is meaningless to speak about one geometry as the *true* geometry. We obtain a statement about physical reality only if in addition to the geometry G of the space its universal field of force F is specified. Only the combination

Only a combination of geometry with coordinative definitions is testable (compare Einstein)

$$G+F$$

is a testable statement.

We can now understand the significance of a decision for Euclidean geometry on the basis of a visual *a priori*. The decision means only the choice of a specific coordinative definition. In our definition of the rigid body we set $F=0$; the statement about the resulting G is then a univocal description of reality. This definition means that in "$G+F$" the second factor is zero. The visual *a priori*, however, sets $G=G_0$. But then the empirical component in the results of measurements is represented by the determination of F; only through the combination

$$G_0+F$$

are the properties of space exhaustively described.

There is nothing wrong with a coordinative definition established on

[1] Generally the force F is a tensor. If $g'_{\mu\nu}$ are the metrical coefficients of the geometry G' and $g_{\mu\nu}$ those of G, the potentials $F_{\mu\nu}$ of the force F are given by

$$g'_{\mu\nu}+F_{\mu\nu}=g_{\mu\nu} \qquad \mu\nu=1,2,3$$

The measuring rods furnish directly the $g'_{\mu\nu}$; the $F_{\mu\nu}$ are the "correction factors" by which the $g'_{\mu\nu}$ are corrected so that $g_{\mu\nu}$ results. The universal force F influencing the measuring rod is usually dependent on the orientation of the measuring rod. About the mathematical limitation of theorem θ cf. §12.

the requirement that a certain kind of geometry is to result from the measurements. We ourselves renounced the simplest form of the coordinative definition, which consists in pointing to a measuring rod; instead we chose a much more complicated coordinative definition in terms of our distinction between universal and differential forces. A coordinative definition can also be introduced by the prescription what the result of the measurements is to be. "The comparison of length is to be performed in such a way that Euclidean geometry will be the result"—this stipulation is a possible form of a coordinative definition. It may be compared to the definition of the meter in terms of the circumference of the earth: "The unit is to be chosen in such a way that 40 million times this length will be equal to the circumference of the earth."

Visual geometry has no relevance for physical geometry

Although it may be admitted that Euclidean geometry is unique in that it can be easily visualized, the theory of the visual *a priori* does not disprove the theory of the relativity of geometry and of the necessity for coordinative definitions of the comparison of length. On the contrary, it is only this theory that can state precisely the epistemological function of visualization: the possibility of visualization is a ground for subjective preference of one particular coordinative definition. But the occurrence of visualization does not imply anything about the space of real objects.

Objection: E is the simplest geometry (compare Poincare)

Reply: in physics we choose the simplest coordinative definition, not the simplest geometry

In this connection another argument in support of the preference for Euclidean geometry is frequently adduced. To be sure, this argument is not related to the problem of visualization, but like the visual *a priori* it attributes a specific epistemological position to Euclidean geometry; therefore we shall consider it here. It is maintained that Euclidean geometry is the *simplest* geometry, and hence physics must choose the coordinative definition $G = G_0$ rather than the coordinative definition $F = 0$. This point of view can be answered as follows: physics is not concerned with the question which *geometry* is simpler, but with the question which *coordinative definition* is simpler. It seems that the coordinative definition $F = 0$ is simpler, because then the expression $G+F$ reduces to G. But even this result is not essential, since in this case simplicity is not a criterion for truth. Simplicity certainly plays an important part in physics, even as a criterion for choosing between physical hypotheses. The significance of simplicity as a means to knowledge will have to be carefully examined in connection with the problem of induction, which does not fall within the scope of this book.

Geometry is concerned solely with the simplicity of a *definition*, and therefore the problem of empirical significance does not arise. It is a mistake to say that Euclidean geometry is "more true" than Einstein's geometry or vice versa, because it leads to simpler metrical relations. We said that Einstein's geometry leads to simpler relations because in it $F = 0$. But we can no more say that Einstein's geometry is "truer" than Euclidean geometry, than we can say that the meter is a "truer" unit of length than the yard. The simpler system is always preferable; the advantage of meters and centimeters over yards and feet is only a matter of economy and has no bearing upon reality. *Properties of reality are discovered only by a combination of the results of measurement with the underlying coordinative definition.* Thus it is a characterization of objective reality that (according to Einstein) a three-dimensional non-Euclidean geometry results in the neighborhood of heavenly bodies, if we define the comparison of length by transported rigid rods. But only the *combination* of the two statements has objective significance. The same state of affairs can therefore be described in different ways. In our example it could just as well be said that in the neighborhood of a heavenly body a universal field of force exists which affects all measuring rods, while the geometry is Euclidean. Both combinations of statements are equally true, as can be seen from the fact that one can be transformed into the other. Similarly, it is just as true to say that the circumference of the earth is 40 million meters as to say that it is 40 thousand kilometers. The significance of this simplicity should not be exaggerated; this kind of simplicity, which we call *descriptive simplicity*, has nothing to do with truth.

Paraphrase of the G+F formula

Taken alone, the statement that a certain geometry holds for space is therefore meaningless. It acquires meaning only if we add the coordinative definition used in the comparison of widely separated lengths. The same rule holds for the geometrical shape of bodies. The sentence "The earth is a sphere" is an incomplete statement, and resembles the statement "This room is seven units long." Both statements say something about objective states of affairs only if the assumed coordinative definitions are added, and both statements must be changed if other coordinative definitions are used. These considerations indicate what is meant by *relativity of geometry*.

Early conventionalism

This conception of the problem of geometry is essentially the result of the work of Riemann, Helmholtz, and Poincaré and is known as *conventionalism*. While Riemann prepared the way for an application of geometry to physical reality by his mathematical formulation of the concept of space, Helmholtz laid the philosophical foundations. In particular, he recognized the connection of the problem of geometry with that of rigid bodies and interpreted correctly the possibility of a visual representation of non-Euclidean spaces (cf. p. 63). It is his merit, furthermore, to have clearly stated that Kant's theory of space is untenable in view of recent mathematical developments.[1] Helmholtz' epistemological lectures must therefore be regarded as the source of modern philosophical knowledge of space.[2] It is Einstein's achievement to have applied the theory of the relativity of geometry to physics. The surprising result was the fact that the world is non-Euclidean, as the theorists of relativity are wont to say; in our language this means: if $F = 0$, the geometry G becomes non-Euclidean. This outcome had not been anticipated, and Helmholtz and Poincaré still believed that the geometry obtained could not be proved to be different from Euclidean geometry. Only Einstein's theory of gravitation predicted the non-Euclidean result which was confirmed by astronomical observations. The deviations from Euclidean geometry, however, are very small and not observable in everyday life.

The error of early conventionalism

Unfortunately, the philosophical discussion of conventionalism, misled by its ill-fitting name, did not always present the epistemological aspect of the problem with sufficient clarity.[3] From conventionalism the consequence was derived that it is impossible to make an objective

[1] The antithesis Kant-Helmholtz has been interpreted by Neo-Kantians (in particular by Riehl, *Kantstudien* 9, p. 261f., less plainly by Görland, *Natorp-Festschrift*, p. 94f) not as a contradiction but as a misunderstanding of Kant by Helmholtz. The same argument has been advanced by Neo-Kantians recently with respect to Einstein's theory. This conception is due to an underestimation of the differences between the points of view, and it would be in the interest of a general clarification if the patent contradiction between the only possible modern philosophy of space and Kant were admitted. Such an admission avoids the danger of an interpretation of Kant's philosophy too vague to retain any concrete content. The author presented his ideas on the subject in "Der gegenwärtige Stand der Relativitätsdiskussion," *Logos* X, 1922, section III, p. 341. Cf. also p. 31. (The English translation of this paper will be included in a forthcoming volume of *Selected Essays* by Hans Reichenbach, to be published by Routledge and Kegan Paul, London.)

[2] Cf. the new edition by Hertz and Schlick, *Helmholtz' Erkenntnistheoretische Schriften*, Berlin 1921.

[3] This is also true of the expositions by Poincaré, to whom we owe the designation of the geometrical axioms as conventions (*Science and Hypothesis*, Dover Publications, Inc. 1952, p. 50) and whose merit it is to have spread the awareness of the definitional character of congruence to a wider audience. He overlooks the possibility of making objective statements about real space in spite of the relativity of geometry and deems it impossible to "discover in geometric empiricism a rational meaning" (*op. cit.*, p. 79). Cf. § 44.

statement about the geometry of physical space, and that we are dealing with subjective arbitrariness only; the concept of geometry of real space was called meaningless. This is a misunderstanding. Although the statement about the geometry is based upon certain arbitrary definitions, the statement itself does not become arbitrary: once the definitions have been formulated, it is determined through objective reality alone which is the actual geometry. Let us use our previous example: although we can define the scale of temperature arbitrarily, the indication of the temperature of a physical object does not become a subjective matter. By selecting a certain scale we can stipulate a certain arbitrary number of degrees of heat for the respective body, but this indication has an objective meaning as soon as the coordinative definition of the scale is added. On the contrary, it is the significance of coordinative definitions to lend an objective meaning to physical measurements. As long as it was not noticed at what points of the metrical system arbitrary definitions occur, all measuring results were undetermined; only by discovering the points of arbitrariness, by identifying them as such and by classifying them as definitions can we obtain objective measuring results in physics. *The objective character of the physical statement is thus shifted to a statement about relations.* A statement about the boiling point of water is no longer regarded as an absolute statement, but as a statement about a relation between the boiling water and the length of the column of mercury. There exists a similar objective statement about the geometry of real space: *it is a statement about a relation between the universe and rigid rods.* The geometry chosen to characterize this relation is only a mode of speech; however, our awareness of the relativity of geometry enables us to formulate the objective character of a statement about the geometry of the physical world as a statement about relations. In this sense we are permitted to speak of *physical geometry*. The description of nature is not stripped of arbitrariness by naive absolutism, but only by recognition and formulation of the points of arbitrariness. The only path to objective knowledge leads through conscious awareness of the role that subjectivity plays in our methods of research.

Physical geometry is not arbitrary, though based on coordinative definitions

§ 9. THE VISUALIZATION OF EUCLIDEAN GEOMETRY

With the result of the foregoing section in mind we turn now to the second question essential to the theory of the visual *a priori* of Euclidean